Master's Thesis

Utility Indifference Prices of Contingent Claims on Nontraded Assets in the Small and Large Claim Limit

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Abstract

This Master's thesis explores approximations of utility indifference prices for a contingent claim written on a nontraded asset in both a small and large position size limit. For position sizes in the small claim limit, optimal hedging strategies are established in a basis risk model. Up to first order, they differ only in the magnitude of a Delta hedge term from the ones derived in the complete market. By this, the deviation of the resulting average utility indifference price from the standard Black-Scholes price becomes negligible. The large position approach is established in a general stochastic factor model. It is shown that as the position size approaches infinity, the utility function's decay rate for large negative wealths is the primary driver of prices. Moreover, prices are studied in the large claim limit, where, in contrast to the small position approach, not only the claim quantity is allowed to vary but also the markets. By the requirement to the markets of becoming asymptotically complete, it will be shown that, depending on the growth rate of the position size, different limiting prices show up, which may differ significantly from the arbitrage-free Black-Scholes price. All the investigations are affirmed by several examples for power and exponential utility.

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Content

Chapter 1

Introduction and Motivation

The problem of pricing claims in an incomplete market framework is one of the main challenges in Mathematical Finance and still extensively researched and studied worldwide. The theory of pricing claims under market completeness goes back to the famous Black-Scholes model developed by Fischer Black and Myron S. Scholes (further evolved by Robert C. Merton) which has been awarded with the Nobel price in the year 1997. This approach uses the core idea of replicating the claim in order to eliminate any risk and from this deriving a unique, arbitrage-free price. Later, this model and its (strong) assumptions (such as constant volatility, log-normal stock returns, continuous paths of the underlying price process, etc.) have been relaxed and led to numerous models, still widely used and accepted in the industry.

Nevertheless, in reality, we do not have complete markets due to several sources of risk, transaction costs, information asymmetry, illiquidity, market regulations – to list just a few reasons. Mathematically speaking, the perfect replication of the claims is not anymore feasible and there is still some unhedgeable risk left. Therefore, there is not anymore a unique price but a whole interval of possible, arbitrage-free, prices.

It is therefore crucial to include the agent's aversion towards this remaining uncertainty into the model and based on this, determine the price which the agent is willing to pay. This leads to the approach of the valuation of claims by utility indifference.

The goal of this thesis is to study the meaning and to get a better understanding of pricing and hedging a European claim $h(Y_T)$ on an asset Y_t that is not easily traded or even cannot be traded due to several impacts such as aggravated market access, high transaction costs or large counter-party positions sizes. To have an example from practice in mind, one could think about employee stock options: these are submitted to employees as part of their wages. They then can exercise the option but are not allowed (for some ex-ante defined time horizon) to trade the received stock. Thus, finding the fair value of such an option, one can simply assume that the respective underlying is not traded. A second classical example is the one of a stock basket. Options written on a basket of stocks (e.g. in a form of a structured product) are treated similarly as the underlying stocks cannot be traded due to its high transaction costs. For pricing and hedging purposes, one then often takes a highly correlated index or exchange traded fund (ETF) into account. Lastly, the disability for trading in Y_t can also arise from illiquidity, as there is an agent in the markets who holds an enormously large position.

Our focus of this investigation lies in the so-called **basis risk model** which assumes that a traded asset S_t as well as the nontraded asset Y_t follow each a geometric Brownian motion, where the respective underlying Brownian motions have correlation ϱ. We then assume that an agent holds a position of q

units of an European claim $h(Y_T)$ written on Y_t and we address the question of pricing and hedging (by trading in S_t) this instrument.
As afore-mentioned, we include the agent's individual risk aversion into the model. In the economic literature, this is usually done by specifying the investor's utility function $U(x)$ – a concave and non-decreasing function representing the measure of preference of a single investor towards wealth and risk, respectively. By the resulting non-linear attitude towards risk, prices become non-linear in the position size, meaning that the unit price for say 10 units of an asset is not equal to the price per unit for 20 assets. Moreover, a very wealthy person does not have the same aversion towards risk as a poor one. Hence, position size as well as (initial) wealth play a crucial role in the later study.
Summarizing, we assume that each investor tries to maximize, by trading in the asset S_t, her expected utility at time $T > 0$, given she has initial wealth x and holds q contracts of the claim $h(Y_T)$.
Following these thoughts and translating into mathematical expressions, our goal is henceforth to study the **average utility indifference price** $p = p_U(x, q; h)$ which is defined through

$$u_U(x - qp, q; h) = u_U(x, q; 0), \tag{1.0.1}$$

where $u_U(x, q; h)$ is called the **value function** and represents the maximal expected utility at time T an investor may achieve by trading in the traded asset S_t and holding q contracts of a claim $h(Y_T)$.
The left-hand side of (1.0.1) is the maximal expected utility an investor may achieve by trading in S_t and holding q contracts of $h(Y_T)$ started from initial wealth $x - qp$ and we require that this coincides with the maximal utility of an investor achieved by just trading in S_t. Accordingly, Equation (1.0.1) asks for the price p such that the investor is indifferent between holding q contracts of the derivative written on the nontraded asset Y_t and trading in S_t or just trading S_t.
It is therefore crucial to have knowledge about the value function $u_U(x, q; h)$ or about the optimizing strategy and by this implicitly about the average utility indifference price p. But in reality, this is very hard and in most cases not even feasible explicitly. Therefore, one has to study approximations of $u_U(x, q; h)$ to get a deeper understanding.
This thesis presents two approaches of tackling this problem:

1) We will study $u_U(x, q; h)$ by letting the position size q converge to 0. This is the so-called **small claim limit**. The case of $q = 0$ has been extensively studied by Merton in [Mer69] for different utility functions and reincorporated by various other authors. We will establish our theory based on [Hen02]. In one of our main results, we are going to see that in the small claim limit, the optimal strategy of investing into S_t, given an agent holds q units of $h(Y_T)$, is obtained by a decline in the Delta hedge term of the optimal strategy derived from the complete market framework, which is an easy consequence of the results obtained by [Mer69]. From this, average utility indifference prices will be established and numerically compared to the standard Black-Scholes prices. An advantage of this approach is that the market is kept constant in the limiting process. But of course, we do not expect that we get proper pricing results for large claims in this setting. Therefore, we also consider the following approach.

2) Alternatively, based on [Rob13], we will examine the value function in a sequence of filtered probability spaces $(\Omega^n, (\mathcal{F}_t^n)_{0\le t\le T}, \mathbb{F}^n, \mathbb{P}^n)$ representing a sequence of markets which will become asymptotically complete in the limit as we let ϱ converge to 1. This is the so-called **large claim limit.** It turns out that it is of high importance that we let the markets also vary. Each market's structure is assumed to be of the basis risk type as mentioned above and we assume that an agent holds q_n (where $q_n \to \infty$)

contracts of the derivative $h(Y_T)$. We then let n converge to infinity and study the limiting behavior of $p^n(x, q; h)$.

This thesis is organized as follows: In Chapter 2, we provide the necessary tools and definitions for the later study. Moreover, we provide a short overview of a so-called duality approach to the initial expected utility maximization problem and we also present results to ensure the lack of arbitrage which will especially play an important role in the large claim limit approach. Then Chapter 3 presents the small claim limit approach while in Chapter 4, we present the large claim limit approach. Both chapters are ended up with examples, where we investigate optimal strategies and utility indifference prices in different market scenarios. These examples include standard options such as European Call and Put options, but also non-standard such as Power options. In Chapter 5, we compare the two approaches in detail, also by the use of examples, and give an extensive conclusion. To the end, in Chapter 6 we provide a short heuristic overview on a generalized semimartingale model and try to see similarities with the approaches presented here.

Chapter 2

Definitions and Setup

In what follows, we provide the main definitions. For the rest of this chapter, we specify our probability space as follows: We let $T > 0$ and set $[0, T]$ to be the finite time horizon. For each n, let $(\Omega^n, (\mathcal{F}_t^n)_{0\leq t\leq T}, \mathbb{F}^n, \mathbb{P}^n)$ denote a filtered probability space where the filtration $\mathbb{F}^n = (\mathcal{F}_t^n)_{0\leq t\leq T}$ satisfies the usual conditions of completeness and right-continuity. Additionally, we assume zero interest rates[1], hence the safe asset is given by $P_t \equiv 1$. Lastly, we assume that the risky and traded asset S_t^n is an $\mathbb{R}$-valued continuous semimartingale.

Further, we assume that the investor holds q_n units of a nontradable, $\mathbb{F}^n$-measurable contingent claim h^n. Note that q_n as well as h^n are allowed to vary by the markets.[2]

We start with one of the central tools in our study.

Definition 2.0.1. A **utility function** $U(x)$ is an increasing and strictly concave function $U \in C^2(\mathbb{R})$ resp. $U \in C^2(\mathbb{R}_{>0})$.[3] We denote the set of utility functions $U \in C^2(\mathbb{R})$ with an exponential-like decay for large negative wealths by

$$\mathcal{U}_\alpha := \left\{ U \text{ utility function} : \lim_{x\to\infty} U(x) = 0 \text{ and } \lim_{x\to-\infty} -\frac{1}{x}\log(-U(x)) = \alpha \right\}.$$

A canonical example of $U \in \mathcal{U}_\alpha$ is given by[4]

$$U_\alpha(x) := -\frac{1}{\alpha}e^{-\alpha x}.$$

Moreover, we denote the set of utility functions with a similar exponential-like behavior for large negative wealths as $U_\alpha \in \mathcal{U}_\alpha$ by

$$\tilde{\mathcal{U}}_\alpha := \left\{ U \in \mathcal{U}_\alpha : 0 < \liminf_{x\to-\infty} \frac{U(x)}{U_\alpha(x)} \leq \limsup_{x\to-\infty} \frac{U(x)}{U_\alpha(x)} < \infty \right\} \subset \mathcal{U}_\alpha.$$

On the first sight, the difference between $\mathcal{U}_\alpha$ and $\tilde{\mathcal{U}}_\alpha$ is not easy to spot. But one could think of a

[1] Clearly, everything could be derived assuming a short rate $r \neq 0$. For simplicity and readability, we forbear to do so.

[2] Of course, h^n cannot vary in any imaginable way, as we require that h^n has to converge in some sense. Hence, we do not mean that h^n varies between e.g. a Put option and a Call option. Rather, we want to emphasize by the superscript n that the value of our claim is dependent on the varying markets, hence on the varying price process.

[3] There is no clear consensus in literature and utility functions are sometimes defined on the whole real line or just on the positive part, depending on the point of interest.

[4] A non-standard example of $U \in \mathcal{U}_\alpha$ is the following: Assume that a wealth manager has N clients, each with canonical exponential utility function U_{α_i}, $1 \leq i \leq N$. Then, under the assumption of fair management, the manager's utility function is given by the weighted average of the different individual utilities, i.e. $U = \sum_{i=1}^N \omega_i U_{\alpha_i}$, where ω_i denotes the proportion of managed wealth of client i with respect to the whole fund. It then can be shown that $U \in U_\alpha$ for $\alpha = \max_i \alpha_i$.

utility function $U(x)$ that satisfies $U(x) = -\frac{1}{x}U_\alpha(x)$ for large negative wealths. It then follows that $U(x) \in \mathcal{U}_\alpha \setminus \tilde{\mathcal{U}}_\alpha$. Lastly, for $p > 1, l > 0$ and $U \in C^2(\mathbb{R})$, we define

$$\mathcal{U}_{p,l} := \left\{ U \text{ utility function: } \lim_{x\to\infty} U(x) = 0 \text{ and } \lim_{x\to-\infty} -\frac{U(x)}{(-x)^p} = \frac{1}{l} \right\},$$

and we call $U \in \mathcal{U}_{p,l}$ a utility function with power-like decay for large negative wealths.
A common example of a utility function supporting only positive wealths is the power utility function

$$U_R(x) := \frac{x^{1-R}}{1-R},$$

for $R > 0$ and $R \neq 1$.

The economical interpretation of utility functions is clear: They are increasing to indicate that agents prefer higher wealths than lower and the concavity indicates the fact that agents are risk-averse.

Example 2.0.1. We provide some examples and properties of above presented utility functions:

1) *Exponential utility function*:
$U_\alpha(x) := -\frac{1}{\alpha}e^{-\alpha x}$ for $\alpha > 0$ and for $x \in \mathbb{R}$. Clearly $U_\alpha(x) \in \mathcal{U}_\alpha$.
A nice feature of this utility function is that it has constant **absolute risk aversion** $\dfrac{-U''_\alpha(x)}{U'_\alpha(x)} = \alpha$.

2) *Power law utility function:*
$U_R(x) := \frac{x^{1-R}}{1-R}$ for $R > 0, R \neq 1$ and for $x \in \mathbb{R}^+$.
A nice feature of this utility function is that is has constant **relative risk aversion** $-x\dfrac{U''_R(x)}{U'_R(x)} = R$.

3) *Utility function with a power-like decay for large negative wealths:*
$U_{p,l}(x) := -\frac{1}{l}x^p - K$ for $x \leq -M < 0$, $K > 0$, $p > 1$ and $l > 0$. Clearly $U_{p,l}(x) \in \mathcal{U}_{p,l}$
One should extend this function in such a way that the concavity property is not violated and such that $U_{p,l}(x) \to 0$ for $x \to \infty$.

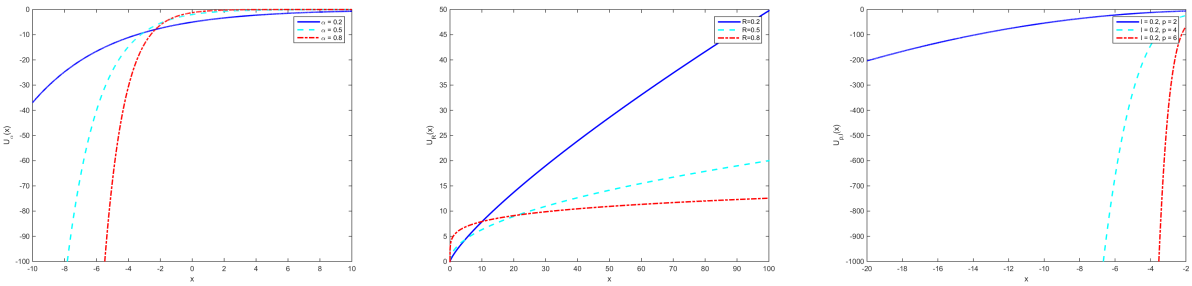
Note. Later, we use utility functions for maximizing expected utility. Expected utility is interpreted *ordinally*, meaning that the magnitude and sign does not matter but rather the order. Therefore it should be clear that in this context, expected utility functions are unique up to linear transformations. Relative and absolute risk aversion are measures that are *invariant* under these transformations. They represent the curvature of the utility function[5] and are also called **Arrow-Pratt** risk aversion coefficients. Put differently, these risk aversion parameters express how much utility an agent gains when she adds (an absolute or relative) amount of wealth to the current wealth. They are therefore local parameters, in the sense that they depend on the current wealth.

Figure 2.0.1 shows examples of utility functions with constant absolute and constant relative risk aversion respectively for different parameters and the behavior of $U_{p,l}(x)$ for $x \leq -2$.

Remark 2.0.1. Let us record a fundamental relationship between the absolute risk aversion α and relative risk aversion R, namely that $\alpha = \frac{R}{x}$. This will be used in the sequel for comparison. As this relation is only local, one has to be careful when using it.

Note. For any utility function presented in Example 2.0.1, it can be shown that it satisfies the conditions known as

[5]It is an *economical* way of representing the curvature. In mathematics, especially in geometry, one has another understanding of curvature.



(a) $U_\alpha(x)$. (b) $U_R(x)$. (c) $U_{p,l}(x)$.

Figure 2.0.1: Example of different utility functions.

1. **Inada Conditions**

$$\lim_{x \to -\infty} U'(x) = \infty \text{ and } \lim_{x \to \infty} U'(x) = 0.$$

2. **Conditions of Reasonable Asymptotic Elasticity**

$$\liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1 \text{ and } \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.$$

The Inada conditions basically ensure that the investor is less risk averse with increasing wealth - and in the limit, the investor's risk aversion is 0, which is of course a plausible assumption as with increasing wealth, investors can bear higher risk. On the other hand, the terms in the Conditions of Reasonable Asymptotic Elasticity can be understood as the ratio between the marginal utility $U'(x)$ and the average utility $\frac{U(x)}{x}$, meaning that these conditions control in some sense the variation of the risk aversion while wealth is varied. For example, they require the marginal utility to be substantially smaller than the average utility when $x \to \infty$ [Sch01, p. 698]. These conditions are well-known and widely used.

Definition 2.0.2. For a utility function U denote by $V : (0, \infty) \to \mathbb{R}$ the **convex conjugate** to U which is given by:

$$V(y) := \sup_{x \in \mathbb{R}} \{U(x) - xy\}.$$

Note. By definition, we have for any $y > 0$

$$U(x) \leq V(y) + xy.$$

By convex analysis, we obtain the following properties to V (see [OŽ09, p.4]):

- $V(y)$ is strictly convex.
- $V(y)$ is continuously differentiable.
- The following relationship between $U(x)$ and $V(y)$ holds true: $(U'(x))^{-1} = -V'(y)$.
- The domain of $V(y)$ resp. $V'(y)$ can be extended in a natural way to $[0, \infty]$ by prescribing $V(0) = U(\infty)$, $V(\infty) = \infty$, $V'(0) = -\infty$, $V'(\infty) = \infty$.

Remark 2.0.2. For the presented utility functions, one easily obtains that

1. *Exponential utility function:*
   $V_\alpha(y) = \frac{y}{\alpha}(\log(y) - 1)$.

2. *Utility function with a power-like decay for large negative wealths:*
$V_p(y) = \frac{p-1}{p}\left(\frac{l}{p}\right)^{\frac{1}{p-1}} y^{\frac{p}{p-1}}$ for $U_p(x) := \frac{1}{l}p^{p-1}x^p - p^2$.

3. *Power law utility function:*
$V_R(y) = \frac{R}{1-R} y^{\frac{R-1}{R}}$.

Note. For $U \in \mathcal{U}_\alpha$, we have that

$$\lim_{y\to\infty} \frac{V(y)}{V_\alpha(y)} = 1, \text{ for } V_\alpha(y) \text{ as above.} \tag{2.0.1}$$

For $U \in \mathcal{U}_{p,l}$, we have that

$$\lim_{y\to\infty} \frac{V(y)}{V_p(y)} = 1, \text{ for } V_p(y) \text{ as above.} \tag{2.0.2}$$

As we are often dealing with (local) martingale measures, let us provide a proper definition:

Definition 2.0.3. We call $\mathcal{M}^n$ the set of all local martingale measures $\mathbb{Q}^n$ which are absolutely continuous with respect to $\mathbb{P}^n$, hence

$$\mathcal{M}^n := \{\mathbb{Q}^n \ll \mathbb{P}^n \text{ on } \mathcal{F}^n : S^n \text{ is a local martingale under } \mathbb{Q}^n\}.$$

In the sequel we are often interested in probability measures μ which are in some sense close to a given probability measure $\mathbb{P}^n$ in the sense that the measure change has a very small impact on the model (e.g. on martingale properties of some processes). For having a tool to measure the degree of dissimilarity, we introduce the notion of relative entropy, which is a measure of departure from a given measure $\mathbb{P}^n$. [FS91][6]

Definition 2.0.4. For any probability measure $\mu \ll \mathbb{P}^n$ on $\mathbb{F}^n$, the **relative entropy**[7] of μ with respect to $\mathbb{P}^n$ is given by

$$H(\mu|\mathbb{P}^n) := \mathbb{E}^{\mathbb{P}^n}\left[\frac{d\mu}{d\mathbb{P}^n}\log\left(\frac{d\mu}{d\mathbb{P}^n}\right)\right].$$

We note that the relative entropy is always nonnegative and of course we have that $H(\mu|\mathbb{P}^n) = 0 \iff \mu = \mathbb{P}^n$. But clearly, this distance is not a metric, as it does fulfill neither symmetry properties nor a triangle inequality.
Therefore, roughly speaking, the distance from μ to $\mathbb{P}^n$ does not coincide with the distance from $\mathbb{P}^n$ to μ.

Remark 2.0.3. Above definition of the relative entropy is only valid under exponential utility. The more general definition of relative entropy would be

$$\mathbb{E}^{\mathbb{P}^n}\left[V\left(\frac{d\mu}{d\mathbb{P}^n}\right)\right],$$

for V being the convex conjugate of U. Unless it is not clear from the context, we will remark which relative entropy we are considering.

Definition 2.0.5. The set of all local martingale measures $\mathbb{Q}^n \in \mathcal{M}^n$ having finite relative entropy H

[6] For the interested reader, we refer to [Ull96] for a detailed explanation on the fact that the relative entropy is a proper and suitable tool to measure the dissimilarity between two measures.

[7] Also called Kullback-Leiler distance.

with respect to $\mathbb{P}^n$ is denoted by

$$\tilde{\mathcal{M}}^n := \{\mathbb{Q}^n \in \mathcal{M}^n : H(\mathbb{Q}^n|\mathbb{P}^n) < \infty\}.$$

Hence this is the set of local martingale measures $\mathbb{Q}^n$ which are, in some sense, closely related to the physical measure $\mathbb{P}^n$.
In the same way, for more general utilities U, we define the set of local martingale measures having finite relative entropy by

$$\tilde{\mathcal{M}}_V^n := \left\{\mathbb{Q}^n \in \mathcal{M}^n : V\left(\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}\right) < \infty\right\}.$$

Lastly, for the utility with a power-like decay for large negative wealths, we set, for $\gamma = \frac{p}{p-1}$ for some $p > 1$

$$\hat{\mathcal{M}}_V^n := \left\{\mathbb{Q}^n \in \mathcal{M}^n : \mathbb{E}^{\mathbb{P}^n}\left[\left(\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}\right)^\gamma\right] < \infty\right\}.$$

Let's turn our attention to trading strategies. As we will work only with geometric Brownian motions in the sequel, we provide here all the definitions based on the fact that we have positive and continuous price processes.

Definition 2.0.6. A **trading strategy** (also called portfolio) π_t^n is any progressively measurable (hence adapted) stochastic process denoting the number of shares. For a price process S_t^n, we denote by

$$V_t^n := P_t + \pi_t^n S_t^n = 1 + \pi_t^n S_t^n$$

the **value** of the portfolio at time t.
The portfolio is called **self-financing** if there is no in- or outflow of capital during any time of trading, i.e. if for all $t \in [0, T]$

$$dV_t^n = \pi_t^n dS_t^n.$$

Hence any change in the value is only due to changes of prices.
A self-financing trading strategy π_t^n is called **admissible** if it is predictable, S_t^n-integrable under $\mathbb{P}^n$ and such that the value process V_t^n is uniformly bounded from below by a constant. We denote the set of all admissible trading strategies by

$$\mathcal{H}^n := \{\pi_t^n \text{ trading strategy } : \pi_t^n \text{ is admissible}\}.$$

The notion of admissibility is to avoid weird trading strategies (such as doubling strategies, where we would possibly need an infinite credit). Based on this, we can now define wealth.

Definition 2.0.7. The **wealth** X_T at time T is given by

$$X_T = X_t + \int_t^T \pi_u^n dS_u^n = X_t + \int_t^T \tilde{\pi}_u^n \frac{dS_u^n}{S_u^n}$$

for a strictly positive semimartingale S_u^n and π_u^n a trading strategy denoting the number of stocks (resp. $\tilde{\pi}_u^n$ denoting the total amount of cash invested in S_t^n). Hence X_T is composed from wealth X_t at time t and the gains resp. losses from trading according to π_u^n in S_u^n between time t and T.

Definition 2.0.8. An **arbitrage portfolio** is a self-financing trading strategy π_t^n with value process $V(0) = 0$ and $V(T) \geq 0$ with $\mathbb{P}^n[V(T) > 0] > 0$. If there is no arbitrage portfolio, we say that the model is **arbitrage-free.**

Definition 2.0.9. A claim h due at time T is **attainable** if there exists an admissible strategy π_t^n which **replicates** h at time T. That is, the value process satisfies $V_T = h$ $\mathbb{P}^n$-a.s.

Definition 2.0.10. We say that a market model is **complete**, if any bounded T-claim is attainable.

At this point, we refer to the two Fundamental Theorems of Asset Pricing, see Theorem A.0.1 and Theorem A.0.2. These connect the notion of no arbitrage and completeness with equivalent martingale measures.

We are now finally able to define the value function by which one of our main focus, the average utility indifference price, is implicitly defined.

Definition 2.0.11. Let U be a utility function. Then for x in the domain of U and $q \in \mathbb{R}$, the **value function** is given by

$$u_U^n(x, q; h^n) := \sup_{\pi^n \in \mathcal{H}^n} \mathbb{E}^{\mathbb{P}^n}[U(x + (\pi^n \cdot S^n)_T + qh^n)]. \tag{2.0.3}$$

As mentioned earlier, the investor's goal is to maximize over all possible strategies (in this case admissible strategies) her expected utility at time T, meaning she wants to maximize the expected wealth which can be decomposed into the initial wealth, the additional (possibly negative) wealth from trading and the gains/losses from holding q contracts of the claim h^n. This is exactly represented in the value function.

Definition 2.0.12. The **average utility indifference price** $p_U^n(x, q; h^n)$ is implicitly given as a solution of

$$u_U^n(x, q; 0) = u_U^n(x - qp_U^n(x, q; h^n), q; h^n). \tag{2.0.4}$$

Hence, $p_U^n(x, q; h^n)$ is the price which an investor with utility function U is prepared to pay per unit of h^n in order to be indifferent between owning and not owning q units of h^n.

Note. We will see later in our study that for a canonical exponential utility function $U_\alpha \in \mathcal{U}_\alpha$, the average utility indifference price $p_{U_\alpha}^n(x, q; h^n)$ is independent of the initial capital x. In reality, it is not always desirable to have this property as one might not assume that investors with different initial wealth have the same attitude towards risk [Hen02, Section 1].

2.1 From the Primal to the Dual Problem

In this section we discuss a possible approach of solving the optimization problem in (2.0.3) to find the value function. There are two well-known ways of tackling the problem: the method of dynamic programming and the duality (or martingale) approach - we will focus on the latter.
To be more rigorous, we record here again the primal problem, that is, each investor tries to maximize over all admissible trading strategies her expected utility of wealth at time T

$$u_{\text{Primal}}^n = \sup_{\pi^n \in \mathcal{H}^n} \mathbb{E}^{\mathbb{P}^n}[U(x + (\pi^n \cdot S^n)_T + qh^n)]. \tag{Primal Problem}$$

Let us state the dual problem to above primal problem, that is

$$u_{\text{Dual}}^n = \inf_{\mu^n \in \text{Cone}(\tilde{\mathcal{M}}_V^n)} \mathbb{E}^{\mathbb{P}^n}\left[V\left(\frac{d\mu^n}{d\mathbb{P}^n}\right) + \frac{d\mu^n}{d\mathbb{P}^n}(x + qh^n)\right], \tag{Dual Problem}$$

where $\text{Cone}(\tilde{\mathcal{M}}^n) := \{y\mathbb{Q}^n : y \geq 0, \mathbb{Q}^n \in \tilde{\mathcal{M}}_V^n\}$.
Using $U(x) \leq V(y) + xy$ for any $y > 0$, it follows that for any equivalent martingale measure $\mathbb{Q}^n \approx \mathbb{P}^n$

$$u^n_{\text{Primal}} \leq \mathbb{E}^{\mathbb{P}^n}\left[V\left(y\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}\right)\right] + \mathbb{E}^{\mathbb{Q}^n}\left[y\left(x + (\pi^n \cdot S^n)_T + qh^n\right)\right] \tag{2.1.1}$$
$$\leq \mathbb{E}^{\mathbb{P}^n}\left[V\left(y\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}\right)\right] + y(x + q\mathbb{E}^{\mathbb{Q}^n}[h^n]) \leq u^n_{\text{Dual}}.$$

But first of all, we notice that the second inequality in (2.1.1) comes from the fact that $(\pi^n \cdot S^n)_T$ is a local martingale under the equivalent martingale measure $\mathbb{Q}^n$ bounded from below, hence a $\mathbb{Q}^n$-supermartingale by Fatou's lemma. As not any supermartingale is bounded from below, we expect that the optimal strategy in the primal problem lies rather in the set $\mathcal{H}^n_{\text{perm}}$ of permissible trading strategies, that is in the set

$$\mathcal{H}^n_{\text{perm}} := \{\pi^n \text{ trading strategy} : (\pi^n \cdot S^n)_T \text{ is a } \mathbb{Q}^n\text{-supermartingale for all } \mathbb{Q}^n \text{ in } \tilde{\mathcal{M}}_V^n\}.$$

We denote the value function given by maximizing over all permissible trading strategies by

$$u^n_{\text{Primal, perm}} := \sup_{\pi^n \in \mathcal{H}^n_{\text{perm}}} \mathbb{E}^{\mathbb{P}^n}[U(x + (\pi^n \cdot S^n)_T + qh^n)]. \tag{Primal Problem'}$$

We easily get that $u^n_{\text{Primal}} \leq u^n_{\text{Primal,perm}}$ as $\mathcal{H}^n \subset \mathcal{H}^n_{\text{perm}}$ due to the fact that every local martingale which is uniformly bounded from below is a supermartingale.
We note that we can rewrite (Dual Problem) as

$$u^n_{\text{Dual}} = \inf_{y \geq 0} \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}_V^n} \left\{\mathbb{E}^{\mathbb{P}^n}\left[V\left(y\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}\right)\right] + y(x + q\mathbb{E}^{\mathbb{Q}^n}[h^n])\right\}.$$

We observe how the (generalized) relative entropy comes into play - it plays a crucial role in the study of the dual problem.
Henceforth, our goal is to find a solution to (Dual Problem) that has minimal/no duality gap, which is the difference in the inequality $u^n_{\text{Primal}} \leq u^n_{\text{Dual}}$.
For this purpose we present in a first step a result which guarantees under certain assumptions, that we do not have to enlarge the set over which the investor maximizes her expected utility, meaning that we have $u^n_{\text{Primal}} = u^n_{\text{Primal,perm}}$ and in a second step that we then even get a duality gap of zero.

Proposition 2.1.1. ([OŽ09, Theorem 1.9])

1) *Assume that the utility function U satisfies the Conditions of Reasonable Asymptotic Elasticity and that $\exists\, x', x'' \in \mathbb{R}$ and $\pi^n \in \mathcal{H}^n$ such that for the claim h^n*

$$x' \leq h^n \leq x'' + (\pi^n \cdot S^n)_T. \tag{2.1.2}$$

Then

$$\tilde{\mathcal{M}}_V^n \neq \emptyset \iff u^n_{\textit{Primal}} < U(\infty).$$

In that case, we have

$$u^n_{\textit{Primal}} = u^n_{\textit{Primal, perm}}, \text{ hence (Primal Problem)} \equiv \text{(Primal Problem')}.$$

2) *Assume that the utility function U satisfies the Conditions of Reasonable Asymptotic Elasticity, that $\tilde{\mathcal{M}}_V^n \neq \emptyset$ and that $\exists\, x', x'' \in \mathbb{R}$ and $(\pi^n)' \in \mathcal{H}_{mg}^n := \{\pi^n \in \mathcal{H}_{perm}^n : (\pi^n \cdot S^n)$ is a $\mathbb{Q}^n$-martingale $\forall\, \mathbb{Q}^n \in \tilde{\mathcal{M}}_V^n\}$ and $(\pi^n)'' \in \mathcal{H}_{perm}^n$ such that*

$$x' + ((\pi^n)' \cdot S^n)_T \leq h^n \leq x'' + ((\pi^n)'' \cdot S^n)_T. \tag{2.1.3}$$

Then for $\tilde{\mathcal{M}}_{equiv}^n := \{\mathbb{Q}^n \approx \mathbb{P}^n$ on $\mathcal{F}^n : S^n$ is a local martingale under $\mathbb{Q}^n$ and $H(\mathbb{Q}^n|\mathbb{P}^n) < \infty\}$ we have

$$\tilde{\mathcal{M}}_{equiv}^n \neq \emptyset \iff \exists \text{ optimal } \pi^{n\star} \in \mathcal{H}_{perm}^n \text{ in (Primal Problem')}.$$

Proof. The complete proof can be found in [OŽ09, Theorem 1.9]. The core idea lies in applying the Lagrange Duality Theorem. □

Note.

- The requirements to h^n in (2.1.2) resp. (2.1.3) are not very restrictive in our study as they are satisfied for example by every bounded claim. They can be understood as an assumption to h^n being sub- and superreplicable.
- Proposition 2.1.1 can be understood as a *version* of the **Fundamental Theorem of Asset Pricing** as it relates the existence of a local martingale measure to the notion of a certain no-arbitrage condition given in terms of finiteness of the maximal utility. This no-arbitrage is sometimes called no nirvana. We refer to Section 2.2.1 for more details.

Next, we want to give conditions that guarantee that (Dual Problem) and (Primal Problem) are equivalent, meaning $u^n_{\text{Primal}} = u^n_{\text{Dual}}$ by this zero duality gap.

Proposition 2.1.2. ([OŽ09, Theorem 1.8]) *Suppose that the utility function U satisfies the Conditions of Reasonable Asymptotic Elasticity and that the claim h^n satisfies (2.1.2) or the weaker (2.1.3). Moreover, assume that $\tilde{\mathcal{M}}_V^n \neq \emptyset$. Then*

1) $u^n_{Primal} = u^n_{Dual} < U(\infty)$, *hence* (Primal Problem) = (Dual Problem).

2) *There exists a $\hat{\mu}^n \in Cone(\tilde{\mathcal{M}}_V^n) \setminus \{0\}$ which is optimal in* (Dual Problem).

If in addition, $\tilde{\mathcal{M}}_{equiv}^n \neq \emptyset$, then

3) $\hat{\mu}^n \in Cone(\tilde{\mathcal{M}}_{equiv}^n) \setminus \{0\}$ *and there exists $(\pi^n)^\star \in \mathcal{H}_{perm}^n$ which is optimal in* (Primal Problem).

Proof. The full proof can be found in [OŽ09, Theorem 1.8]. □

Conclusion:
As long as U satisfies the Conditions of Reasonable Asymptotic Elasticity and h^n is assumed to be both super- and subreplicable, then (Primal Problem) is equivalent to (Dual Problem) as soon as there exists at least one equivalent local martingale measure $\mathbb{Q}^n$ having finite (generalized) relative entropy with respect to $\mathbb{P}^n$, i.e. as soon as $\tilde{\mathcal{M}}_V^n \neq \emptyset$.

We are now able to properly formulate our model.

2.2 Model Formulation

We address the question of pricing and hedging European claims $h^n(Y_T)$ written on some nontraded asset Y_t. The fact of disability of trading in Y_t can originate in different reasons - we have given some in the introduction. Especially for hedging purposes, what practitioners do is to take a closely related and traded asset S_t^n into account for trying to hedge their exposure optimally. This will now be translated into a mathematical model.

We consider a stochastic basis $(\Omega^n, (\mathcal{F}_t^n)_{0\leq t\leq T}, \mathbb{F}^n, \mathbb{P}^n)$ which supports two correlated Brownian motions B_t, Z_t. For each $n \in \mathbb{N}$, we consider the assets S_t^n, Y_t given by the following stochastic differential equations

$$\frac{dS_t^n}{S_t^n} = \mu(Y_t)dt + \sigma(Y_t)dB_t, \quad S_0^n \qquad\qquad dY_t = \nu(Y_t)dt + \eta(Y_t)dZ_t, \quad Y_0. \tag{2.2.1}$$

It is convenient to express Z_t as a linear combination of two independent Brownian motions B_t, W_t, namely

$$Z_t = \varrho_n B_t + \sqrt{1-\varrho_n^2}W_t,$$

for $\varrho_n \in [-1, 1]$.

There are several ways to interpret Y_t. For example, Y_t can be seen as describing a certain factor of the asset S_t^n (e.g. micro-/macroeconomic factor such as taxes, inflation, etc.). The model is thus called **stochastic factor model.** As a special case, we can interpret Y_t as a nontraded asset but on which claims may be written. We are then exactly in the framework of a nontraded asset Y_t and a traded reference asset S_t^n as addressed in the introduction. For simplicity, we then assume that S_t as well as Y_t are given by a geometric Brownian motions (i.e. $\nu(Y_t) = \nu Y_t$, $\eta(Y_t) = \eta Y_t.$, $\mu(Y_t) = \mu$, $\sigma(Y_t) = \sigma$). Hence, all our price processes are continuous.

When trading in S_t^n for (proxy) hedging a position in $h^n(Y_T)$, the crucial and very obvious fact is, that, as long as $\varrho_n \neq \pm 1$, there is still some unhedgeable basis risk left. This gives the name **basis risk model.** In Chapter 3 we discuss the basis risk model in more detail, while Chapter 4 gives a detailed insight in the general stochastic factor model.

We then study the average utility indifference price $p_u^n(x, q_n; h^n)$ given by the utility indifference criterion

$$u_U^n(x, q_n; 0) = u_U^n(x - q_n p_U^n(x, q_n; h^n), q_n; h^n).$$

As this equation cannot be solved explicitly, we are interested in its limit approximations as $q_n \to 0$ (small claim limit, Chapter 3) resp. $q_n \to \infty$ (large claim limit, Chapter 4).

For this, we apply the duality approach as introduced in the preceding section.

Switching the initial primal problem of finding the optimizing strategy $\pi^{n\star} \in \mathcal{H}^n$ in

$$u_{\text{Primal}}^n = \sup_{\pi^n\in\mathcal{H}^n} \mathbb{E}^{\mathbb{P}^n}[U(x + (\pi^n \cdot S^n)_T + qh^n)]$$

to the dual problem

$$u_{\text{Dual}}^n = \inf_{y\geq 0} \inf_{\mathbb{Q}^n\in\tilde{\mathcal{M}}^n} \left\{ \mathbb{E}^{\mathbb{P}^n}\left[V\left(y\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}\right)\right] + y(x + q\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)]) \right\}$$

leads us to an optimization over a set of (probability) measures.

It will turn out later in our study that it is a good idea for finding the optimizing probability measure in

(Dual Problem) to consider the so-called **minimal martingale measure** $\mathbb{Q}_{\min}$ introduced by [FS91] - a measure that became fairly popular recently when dealing with continuous price processes. We present here an intuitive construction based on [Sch99b] and [FS10]
Consider a continuous adapted process X_t with Doob-Meyer decomposition ([Pro04, Section III, Theorem 6])

$$X_t = X_0 + M_t + A_t,$$

where M_t is a (square integrable) local $\mathbb{P}$-martingale and A_t a predictable process of bounded variation of the form

$$A_t = \int_0^t \lambda_s d\langle M \rangle_s,$$

such that for all $t \in [0, T]$, the **mean-variance trade-off process** satisfies the following integrability condition

$$\Lambda := \frac{1}{2} \int_0^t \lambda_s^2 d\langle M \rangle_s < \infty.$$

Then $\mathbb{Q}_{\min}$ defined through

$$\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}^n} := \mathcal{E}\left(- \int \lambda dM \right)_T$$

is the unique equivalent local martingale measure for X_t with the property that all square-integrable $\mathbb{P}^n$-martingales M_t' strongly $\mathbb{P}^n$-orthogonal to M_t (i.e. $M_t M_t'$ is a $\mathbb{P}$-martingale) are also $\mathbb{Q}_{\min}$-martingales. Roughly speaking, $\mathbb{Q}_{\min}$ is the equivalent (local) martingale measure that preserves the martingale structure as far as possible under the constraint of turning M_t into a martingale.
The name of $\mathbb{Q}_{\min}$ has its origin as it minimizes the reverse relative entropy $H(\mathbb{P}^n|\cdot)$ over all equivalent local martingale measures $\mathbb{Q}$ for X (this holds only under the assumption of continuous paths of X_t). [Sch99b]. Note that as we mentioned in Definition 2.0.4, the (generalized) relative entropy does not fulfill a symmetry property.
In contrast to that, the **minimal entropy martingale measure** is the measure $\mathbb{Q}^n$ that minimizes $H(\cdot|\mathbb{P}^n)$ over all equivalent local martingale measures $\mathbb{Q}^n \sim \mathbb{P}^n$. [Sch10]
By [Sch10, p.3], we even get that the minimal entropy martingale measure coincides with the minimal martingale measure if the price process S_t is continuous and the mean-variance process has constant expectation over all equivalent local martingale measures for S_t.
In the classical basis risk model

$$\frac{dS_t^n}{S_t^n} = \mu dt + \sigma dB_t, \quad S_0 \qquad\qquad \frac{dY_t}{Y_t} = \nu dt + \eta \left(\varrho_n dB_t + \sqrt{1-\varrho_n^2} dW_t \right), \quad Y_0,$$

the minimal martingale measure $\mathbb{Q}_{\min}$ is easily obtained and given by

$$\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}^n} = \mathcal{E}\left(-\frac{\mu}{\sigma} \cdot B \right)_T.$$

In this setting, $\mathbb{Q}_{\min}$ coincides with the natural suggestion of an equivalent martingale measure provided by Girsanov's Theorem. Moreover, under $\mathbb{Q}_{\min}$, it follows by Girsanov's Theorem that $B_t^{\min} := B_t + \frac{\mu}{\sigma} t$ as well as $W_t^{\min} := W_t$ are Brownian motions.

Lastly, the relative entropy of $\mathbb{Q}_{\min}$ with respect to $\mathbb{P}^n$ in the case of exponential utility is given by

$$\begin{aligned} H(\mathbb{Q}_{\min}|\mathbb{P}^n) &= \mathbb{E}^{\mathbb{P}^n}\left[\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}^n}\log\left(\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}^n}\right)\right] \\ &= \mathbb{E}^{\mathbb{P}^n}\left[\exp\left(-\frac{\mu}{\sigma}B_t+\frac{\mu^2}{2\sigma^2}T\right)\left(-\frac{\mu}{\sigma}B_t+\frac{\mu^2}{2\sigma^2}T\right)\right] \\ &< \infty. \end{aligned}$$

Hence $\tilde{\mathcal{M}}^n \neq \emptyset$ and (Primal Problem) = (Dual Problem).

Having defined or model, we want to guarantee that no arbitrage can occur in a fixed market as well as in the limiting process.

2.2.1 Exclusion of Arbitrage Opportunities

We know exactly the conditions to avoid risk-less profit (see Theorem A.0.1) - the existence of at least one local martingale measure. But in an incomplete market framework we do not have anymore a unique pricing measure $\mathbb{Q}^n$ but rather a set of possible pricing measures. However, we consider the set of **viable prices** $\{\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] : \mathbb{Q}^n \text{ is an equivalent martingale measure for } S^n\}$. Viable prices do not include arbitrage opportunities due to the Fundamental Theorem of Asset Pricing, but they give no insight and information about optimal hedging strategies.

The next proposition gives us for the interval of possible prices (sub-/super-)replicating trading strategies for the claim h^n. In literature, this result is therefore also known as **superreplication theorem**.

Proposition 2.2.1. (following [Kal09, Theorem 2] and [DS06, Theorem 2.4.1]) *Let $h = h(Y_T)$ denote the payoff of an (European) claim on a nontraded asset Y_t. Moreover, let S_t^n be the reference traded asset. Then*

$$\begin{aligned} p_{high} := \min\{p \in \mathbb{R} :\ & \textit{There exists some self-financing strategy } \pi \textit{ with initial value} \\ & V_0(\pi) = p \textit{ and terminal value } V_T(\pi) \geq h\} \\ = \sup\{&\mathbb{E}^{\mathbb{Q}^n}[h(Y_T)] : \mathbb{Q}^n \textit{ is an equivalent martingale measure for } S_t^n\} \end{aligned}$$

and

$$\begin{aligned} p_{low} := \max\{p \in \mathbb{R} :\ & \textit{There exists some self-financing strategy } \pi \textit{ with initial value} \\ & V_0(\pi) = p \textit{ and terminal value } V_T(\pi) \leq h\} \\ = \inf\{&\mathbb{E}^{\mathbb{Q}^n}[h(Y_T)] : \mathbb{Q}^n \textit{ is an equivalent martingale measure for } S_t^n\}. \end{aligned}$$

Then the interval of arbitrage-free prices is given by

$$I(h) = [p_{low}, p_{high}].$$

We call p_{low} **subreplication price** and p_{high} **superreplication price**. The intuition behind the arbitrage-free prices is that if someone offers an investor a price $p > p_{\text{high}}$, then she can follow a self-financing strategy with terminal value $V_T(\pi) \geq h$, hence the investor has no risk of losing any money.

Of course, any viable price lies in this interval.

In our notation, the interval of arbitrage-free prices reduces to

$$I(h) = \left[\inf_{\mathbb{Q}^n \in \mathcal{M}^n} \mathbb{E}^{\mathbb{Q}^n}[h(Y_T)], \sup_{\mathbb{Q}^n \in \mathcal{M}^n} \mathbb{E}^{\mathbb{Q}^n}[h(Y_T)] \right].$$

For the large claim approach we also want to guarantee that, as the markets become asymptotically complete, there will be no 'infinite utility'. This is sometimes also called 'no nirvana'.

Lemma 2.2.1. ([Rob13, Proposition 6.1]) *Let $\alpha > 0, p > 1, l > 0$ and $x \in \mathbb{R}$. Then under the assumption that $\tilde{\mathcal{M}}^n \neq \emptyset$ for each n and $\limsup_{n\to\infty} \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} H(\mathbb{Q}^n|\mathbb{P}^n) < \infty$ it follows that for $U \in \mathcal{U}_\alpha$*

$$\limsup_{n\to\infty} u_U^n(x, q; 0) < U(\infty) = 0.$$

Assuming that we have $\hat{\mathcal{M}}_V^n \neq \emptyset$ for all n and that $\limsup_{n\to\infty} \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}_V^n} \mathbb{E}^{\mathbb{P}^n}[(\frac{d\mathbb{Q}^n}{d\mathbb{P}^n})^\gamma] < \infty$, it follows for $U \in \mathcal{U}_{p,l}$

$$\limsup_{n\to\infty} u_U^n(x, q; 0) < U(\infty) = 0.$$

Remark 2.2.1. This result ensures no nirvana in the limiting process. If we had $\limsup_{n\to\infty} u_U^n(x, q; 0) > U(\infty)$, then there would exist a subsequence $\{n_k\}_k$ such that $u_U^{n_k}(x, q; 0) \geq U(\infty)$ for k large enough. By starting with initial wealth x, an investor could then follow this strategy, increase her expected utility and end up with a higher expected utility than the one corresponding to infinite wealth. Such cases should be of course excluded.

Proof. We consider the first statement:
By assumption, we have that there exists a sequence of measures $\mathbb{Q}_1^n \in \tilde{\mathcal{M}}^n$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}^n} \left[V_\alpha \left(\frac{d\mathbb{Q}_1^n}{d\mathbb{P}^n} \right) \right] \leq C. \tag{2.2.2}$$

This implies that $Z_1^n := \frac{d\mathbb{Q}_1^n}{d\mathbb{P}^n}$ is uniformly integrable with respect to $\mathbb{P}^n$,
i.e. $\lim_{\lambda\to\infty} \sup_n \mathbb{E}^{\mathbb{P}^n}[Z_1^n \mathbf{1}_{Z_1^n \geq \lambda}] = 0$. For $x \in \mathbb{R}$, it then follows that

$$u_U^n(x, q; 0) \leq \inf_{y>0} \left(\mathbb{E}^{\mathbb{P}^n}[V(yZ_1^n) + xy] \right).$$

Now we use a fact that we will later prove in Lemma 4.1.2: For a random variable Y with $\mathbb{E}^{\mathbb{P}^n}[Y] = 1$ such that $\mathbb{E}^{\mathbb{P}^n}[V(Y)] < \infty$, we have that the map $y \mapsto \mathbb{E}^{\mathbb{P}^n}[V(yY)]$ is differentiable with surjective derivative $\mathbb{E}^{\mathbb{P}^n}[YV'(yY)]$. In this setting, we set $Y = Z_1^n$ and get the existence of a unique $y_n > 0$ which solves above minimization problem as the map $y \mapsto \mathbb{E}^{\mathbb{P}^n}[V(yZ_1^n)]$ is differentiable and V convex.
By this we get the first order condition $x \overset{!}{=} -\mathbb{E}^{\mathbb{P}^n}[Z_1^n V'(y_n Z_1^n)]$.
Assume for the moment being that

$$\liminf_{n\to\infty} y_n > 0. \tag{2.2.3}$$

Then

$$u_U^n(x, q; 0) \leq \mathbb{E}^{\mathbb{P}^n}[V(y_n Z_1^n)] + xy_n = -\mathbb{E}^{\mathbb{P}^n}[(y_n Z_1^n V'(y_n Z_1^n) - V(y_n Z_1^n)].$$

Define $f(z) := zV'(z) - V(z)$. We observe that $f'(z) = zV''(z) > 0$ for $y > 0$ and that $\lim_{z\to 0} f(z) = 0$ as $U(\infty) = 0$. Moreover, f is increasing and non-negative. Take $\delta > 0$ such that, in view of (2.2.3), we have that $y_n \geq \delta$ for n large enough. This gives

$$u_U^n(x,q;0) \leq -\mathbb{E}^{\mathbb{P}^n}[f(y_n Z_1^n)] \leq -\mathbb{E}^{\mathbb{P}^n}[f(\delta Z_1^n)] \leq 0.$$

In order to end up with a with a contradiction, assume for the moment being the existence of a sequence such that $\lim_{n\to\infty} u_U^n(x,q;0) = 0$. We then get that $\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}^n}[f(\delta Z_1^n)] = 0$ and hence for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}^n[Z_1^n \geq \varepsilon] = 0. \tag{2.2.4}$$

Now fix $\varepsilon > 0$ and choose λ large enough that $\sup_n \mathbb{E}^{\mathbb{P}^n}[Z_1^n \mathbf{1}_{Z_1^n \geq \lambda}] \leq \varepsilon$. Since $Z_1^n \in \tilde{\mathcal{M}}^n$, we get

$$1 = \mathbb{E}^{\mathbb{P}^n}[Z_1^n] = \mathbb{E}^{\mathbb{P}^n}[Z_1^n(\mathbf{1}_{Z_1^n \leq \varepsilon} + \mathbf{1}_{\varepsilon < Z_1^n < \lambda} + \mathbf{1}_{Z_1^n \geq \lambda})] \leq \varepsilon + \lambda \mathbb{P}^n[Z_1^n > \varepsilon] + \varepsilon.$$

If we pass to the limit and let $n \to \infty$ and then $\varepsilon \to 0$, we get a contradiction, hence (2.2.4) can not be true and thus $\lim_{n\to\infty} u_U^n(x,q;0) < 0$.
To complete the proof, we are left with checking the assumption in (2.2.3): Again ending up with a contradiction, assume for the moment being that there exists a sequence y_n such that $\lim_{n\to\infty} y_n = 0$. Let $(M_n)_n$ be another sequence with $\lim_{n\to\infty} M_n = \infty$ and $\lim_{n\to\infty} y_n M_n = 0$. Choose n large enough such that $y_n < 1$ which yields by the convexity of V

$$-x \leq V'(y_n M_n)\mathbb{E}^{\mathbb{P}^n}[Z_1^n \mathbf{1}_{Z_1^n \leq M_n}] + \mathbb{E}^{\mathbb{P}^n}[Z_1^n V'(Z_1^n)\mathbf{1}_{Z_1^n > M_n}]. \tag{2.2.5}$$

We have that $\lim_{n\to\infty} V'(y_n M_n)\mathbb{E}^{\mathbb{P}^n}[Z_1^n \mathbf{1}_{Z_1^n \leq M_n}] = \mathbb{E}^{\mathbb{P}^n}[Z_1^n] \lim_{n\to\infty} V'(y_n M_n) = -\infty$.
From [OŽ09, Assumption 1.2] and [Sch01, Proposition 4.1, Corollary 4.2], we have the existence of a constant C such that $z|V'(z)| \leq CV(z)$ for $z > 0$. Moreover, as $M_n \to \infty$, we get that for any $\varepsilon > 0$, by the property in (2.0.1):

$$V(z)\mathbf{1}_{z \geq M_n} \leq (1+\varepsilon)\left(V_\alpha(z) + \frac{1}{\alpha}\right). \tag{2.2.6}$$

By (2.2.2), we therefore obtain for some large K, that

$$\limsup_{n\to\infty} \mathbb{E}^{\mathbb{P}^n}\left[Z_1^n V'(Z_1^n y_n)\mathbf{1}_{Z_1^n > M_n}\right] \leq K$$

holds true, which contradicts (2.2.5).
For the second statement, we can proceed in the very same way. □

Conclusion:
By the first Fundamental Theorem of Asset Pricing, we get a family of viable prices for a contingent claim h. Then, by the superreplication theorem, we get (at least the existence) self-financing trading strategies that sub- resp. superreplicate the claim. Lastly, the condition of no nirvana, i.e. no asymptotic arbitrage opportunities, when we let the markets vary, is given by $\limsup_{n\to\infty} u_U^n(x,q;0) < U(\infty)$.

Chapter 3

Small Claim Limit Approach

In this chapter, we present the small claim limit approach for deriving an approximation of $u_U(x, q; h)$ near $q = 0$. Our main reference for this approach is [Hen02]: 'Valuation of Claims on nontraded assets using Utility Maximization', *Mathematical Finance*, Vol. 12, No. 4 (October 2002), p: 351 - 373.
Starting with the classical Black-Scholes-Merton model, where we have a traded and risky asset S_t and a safe bank account P_t and the corresponding Black-Scholes-Merton wealth problem with one asset, we include a second, nontraded asset Y_t, which is, in some sense, correlated to the traded asset S_t and on which a claim h is written. This leads us to the basis risk model.
An investor is then assumed to hold q units of the European claim $h(Y_T)$, where T is the endpoint of our finite time horizon $[0, T]$. Under market incompleteness, we analyze the value function $u_U(x, q; h)$ for both the exponential and power law utility function in a neighborhood of $q = 0$. By this, we will show in our main result that a first order approximation of the optimal strategy in the small claim limit is obtained by a decline in the Delta hedge term of the optimal strategy derived from the complete market framework (where the assets S^n and Y are perfectly correlated, i.e. $\varrho = 1$). Having this established, we then find the value function and by this average utility indifference prices.
A nice feature of this approach is that it provides us directly with hedging strategies.

3.1 The Classical Black-Scholes-Merton Model

We work on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{F}, \mathbb{P})$ for some finite time horizon $T > 0$. We consider the well-known **Black-Scholes-Merton** model without transaction costs where we are given a risky asset S_t following a geometric Brownian motion with volatility σ and drift μ and a safe bank account P_t satisfying

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad S_0 \qquad\qquad dP_t = rP_t dt, \quad P_0 = 1, \tag{3.1.1}$$

where $\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$ and B_t is a standard $\mathbb{P}$-Brownian motion and $r \geq 0$ is the risk-free rate. We still assume that $r = 0$. By the specification of the dynamics, the distribution of $\log(S_t)$ is assumed to be Gaussian under the physical measure $\mathbb{P}$ as S_t is given by

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right).$$

The challenge for the investor is now to find an optimal strategy $\tilde{\pi}$ of investing into the risky asset S_t

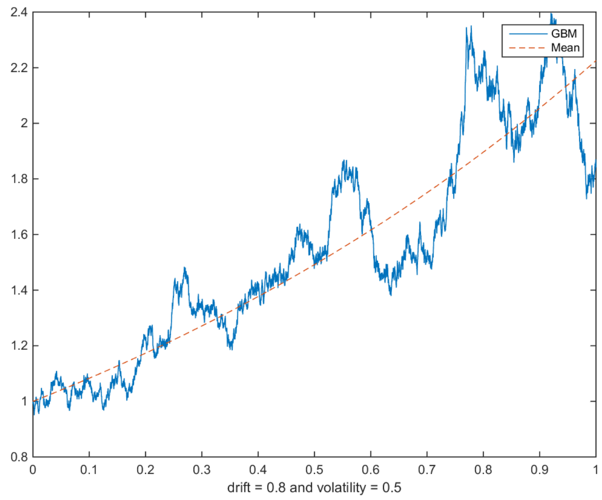


Figure 3.1.1: Trajectory of a geometric Brownian motion.

and the bank account $P_t \equiv 1$, given she has an initial wealth of $x > 0$. Here, 'optimal' is meant in the sense of maximizing her expected utility at time T, i.e. finding the strategy $\tilde{\pi} \in \mathcal{H}$ that maximizes

$$\mathbb{E}^{\mathbb{P}}\left[U\left(x + \int_0^T \tilde{\pi}_t \frac{dS_t}{S_t}\right)\right],$$

given S_t follows a geometric Brownian motion as described above.
Merton showed in his paper [Mer69, Section IV] that for a power law utility function $U_R = \frac{x^{1-R}}{1-R}$ (and without any contingent claim), the optimal proportion of wealth invested in the risky asset (which we denote by $\tilde{H}_t = \frac{\tilde{\pi}_t}{X_t}$) at time $t \leq T$ is given by

$$\tilde{H}_t^\star = \frac{\mu}{\sigma^2 R}.$$

Hence the optimal strategy in this setting consists of an over time constant fraction of the current wealth invested into the risky asset S_t. This is a consequence of the constant relative risk aversion of the power utility. It is of high importance to point out that this fraction is independent of X_t, the current wealth. The corresponding value function is given by

$$\begin{aligned} u_{U_R}(x, q; 0) = \sup_{\tilde{\pi} \in \mathcal{H}} \mathbb{E}^{\mathbb{P}}\left[U_R(X_T)\right] &= \frac{x^{1-R}}{1-R}\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{\mu^2}{\sigma^2 R}T - \frac{\mu^2}{2\sigma^2 R^2}T + \frac{\mu}{\sigma R}B_T\right)^{1-R}\right] \\ &= \frac{x^{1-R}}{1-R}\exp\left(\frac{1}{2}\frac{\mu^2}{\sigma^2}\frac{1-R}{R}T\right). \end{aligned}$$

For an exponential utility function $U_\alpha(x) = -\frac{1}{\alpha}e^{-\alpha x}$, [Mer69, Section IX] shows that the optimal proportion $\tilde{H}_t^\star$ of wealth invested in the risky asset is no longer independent of X_t and we have

$$\tilde{\pi}_t^\star = \frac{\mu}{\sigma^2 \alpha}.$$

Also here, this is a consequence of the constant absolute risk aversion property of the exponential utility function. Hence the optimal strategy of investing into S_t consists of a constant over time risky position (and not anymore a constant proportion as it was the case under power utility). We then obtain the value function which is given by

$$\begin{aligned} u_{U_\alpha}(x,q;0) &= -\frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\alpha\left(x+\int_0^T \tilde{\pi}_t^\star \frac{dS_t}{S_t}\right)\right)\right] \\ &= -\frac{1}{\alpha}\exp(-\alpha x)\exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma^2}T\right). \end{aligned}$$

We remark that in both cases, the optimal strategy of investing into S_t is given by the mean-variance ratio rescaled by the absolute risk aversion.

3.2 The Classical Black-Scholes-Merton Model with an Additional Nontraded Asset

We now consider the classical Black-Scholes-Merton model introduced in Section 3.1, where we additionally include a nontraded asset Y_t following a (correlated) geometric Brownian motion. This leads us to the basis risk model:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad S_0 \qquad\qquad \frac{dY_t}{Y_t} = \nu dt + \eta dZ_t, \quad Y_0, \tag{3.2.1}$$

where $\mu, \nu \in \mathbb{R}$ and $\sigma, \eta \in \mathbb{R}^+$ denote the drift resp. volatility of the associated geometric Brownian motion. Moreover, B_t, Z_t are two Brownian motions with correlation $\varrho \in [-1,1]$ and we set

$$Z_t = \varrho B_t + \sqrt{1-\varrho^2}W_t \tag{3.2.2}$$

for two independent $\mathbb{P}$-Brownian motions B_t, W_t. This gives that $d\langle B, Z\rangle_t = \varrho dt$.
The goal of the investor still is to maximize her expected utility at time T by trading in S_t and, additionally, holding q contracts of $h(Y_T)$, meaning that she wants to maximize in the sense of the afore presented primal problem, that is, finding the maximizing strategy $\tilde{\pi}^\star \in \mathcal{H}$ in

$$u_U(x,q;h) = \sup_{\tilde{\pi}\in\mathcal{H}} \mathbb{E}^{\mathbb{P}}\left[U\left(x+\int_0^T \tilde{\pi}_t \frac{dS_t}{S_t} + qh(Y_T)\right)\right]. \tag{3.2.3}$$

To avoid computational issues, it turns out that the following assumption will play a crucial role in our study:

Assumption 1. We assume that for q and h one of the following assertions hold true:

1. $0 \leq h \leq B$ for some constant $B > 0$ and $q \in \mathbb{R}$.
   Example: long/short position in a Put option.

2. $h \geq 0$ but not bounded from above and $q \in \mathbb{R}^+$.
   Example: long position in a Call option.

The attentive reader notices that there is one type of the four standard option types, namely the position of a 'short Call', which does not fulfill above conditions to q and h and hence will be excluded in what follows. The reason for this exclusion is that the value function becomes identical to minus infinity for

power law and exponential utilities. Indeed, we have that Y_t is the product of a term measurable with respect to the filtration generated by B_t and a random (and independent to B_t) part $e^{\eta\sqrt{1-\varrho^2}W_t}$, from which the unhedgeable risk and the market incompleteness arises. Clearly, this interrupting term is unbounded from above.
Hence, assuming $|\varrho| < 1$, this term has an impact on the behavior of the value function $u_U(x, q; h)$: Moreover, we have for the short Call option $h(Y_T) = -(Y_T - K)^+$ that with positive probability

$$X_T + qh(Y_T) < 0,$$

and hence wealth may become negative.
As the power utility function $U_R(x) = -\infty$[8] for $x \in \mathbb{R}^-$, we have that

$$u_{U_R}(X_t, q; h) = -\infty \text{ for } 0 \leq t \leq T.$$

A similar problem arises if we consider exponential utility, as

$$\mathbb{E}^{\mathbb{P}}[U_\alpha(-(S_T - K)^+)] = -\frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}[\exp(\alpha(S_T - K))\mathbf{1}_{S_T \geq K}] = +\infty,$$

as $\mathbb{E}^{\mathbb{P}}\left[e^{e^N}\right] = +\infty$ for a normal random variable N and by noticing that $\mathbb{E}^P[\exp(\alpha(S_T - K))\mathbf{1}_{S_T<K}] \leq 1$. Hence, we are not able to price short Call options[9] in this model. This is clearly a shortcoming of this approach.

Turning our attention to the dual problem

$$u_{\text{Dual}} = \inf_{y \geq 0} \inf_{\mathbb{Q} \in \tilde{\mathcal{M}}} \mathbb{E}^{\mathbb{P}}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right) + y\frac{d\mathbb{Q}}{d\mathbb{P}} + qh\right],$$

we consider the minimal martingale measure

$$\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}} = \mathcal{E}\left(-\frac{\mu}{\sigma} \cdot B\right)_T.$$

Under this measure, S_t is clearly a martingale as

$$\frac{dS_t}{S_t} = \sigma dB_t^{\min},$$

where $B_t^{\min} = B_t + \frac{\mu}{\sigma}t$ is a $\mathbb{Q}_{\min}$-Brownian motion. Moreover, $W_t^{\min} = W_t$ is also a $\mathbb{Q}_{\min}$-Brownian motion. Hence we have for Y_t that

$$\begin{aligned}\frac{dY_t}{Y_t} &= \nu dt + \eta\varrho dB_t + \eta\sqrt{1-\varrho^2}dW_t \\ &= \left(\nu - \frac{\eta\varrho\mu}{\sigma}\right)dt + \frac{\eta\varrho}{\sigma}(\sigma dB_t + \mu dt) + \eta\sqrt{1-\varrho^2}dW_t,\end{aligned} \tag{3.2.4}$$

where the last two terms are $\mathbb{Q}_{\min}$-martingales and $\delta := \nu - \frac{\mu\varrho\eta}{\sigma}$ is its drift under $\mathbb{Q}_{\min}$. If the market is complete, then there exists a unique equivalent martingale measure and the minimization reduces to

[8] To be exact, we defined the power utility function $U_R(x)$ for $x \in \mathbb{R}_0^+$. Here, we think of it as an extension to $\mathbb{R}$ by setting $U_R(x) = -\infty$ for $x \in \mathbb{R}^-$.

[9] Clearly, in this setting of utility maximization, it is not sufficient to just compute the prices of a long position in some option and then change the sign to get the prices of the respective short position due to the concavity of the investor's individual utility and hence risk aversion.

a minimization over $y \geq 0$. Otherwise, under any market incompleteness, above minimization is not that easy and straightforward anymore.

3.2.1 Complete Basis Risk Model

If $\varrho = 1$, then the two Brownian motions B_t and Z_t are perfectly correlated, meaning that $B_t \overset{\text{law}}{=} Z_t$, hence all the risk and uncertainty arises from the same source, say B_t.
In this setting, we are able to price and hedge in the known ways, as there is one unique martingale measure and hence by the two Fundamental Theorems of Asset Pricing (see Theorem A.0.1 and Theorem A.0.2) the market model is arbitrage-free and even complete. Indeed, we have under the physical measure $\mathbb{P}$

$$\frac{dY_t}{Y_t} = \nu dt + \eta dB_t = \frac{\eta}{\sigma}\frac{dS_t}{S_t} + \left(\nu - \frac{\mu\eta}{\sigma}\right) dt.$$

Clearly, S_t, the traded risky asset, has to be a martingale under the equivalent martingale measure, which we denote by $\bar{\mathbb{Q}}$. We get by Girsanov's Theorem that

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} = \mathcal{E}\left(-\frac{\mu}{\sigma} \cdot B\right)_T .$$

But of course, as we only have one source of risk, we must have

$$\nu = \frac{\mu\eta}{\sigma} \tag{3.2.5}$$

to ensure the lack of arbitrage opportunities. Thus $\bar{\mathbb{Q}}$ has Radon-Nikodym derivative of the form

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} = \mathcal{E}\left(-\frac{\nu}{\eta} \cdot B\right)_T = \mathcal{E}\left(-\frac{\mu}{\sigma} \cdot B\right)_T \left(= \frac{d\mathbb{Q}_{\min}}{d\mathbb{P}}\right). \tag{3.2.6}$$

Turning our attention to the maximization problem in (3.2.3). We consider a new wealth variable

$$\bar{X}_t := X_t + q\bar{C}_t, \text{ for } \bar{C}_t := \mathbb{E}^{\bar{\mathbb{Q}}}\left[h(Y_T)|\mathcal{F}_t\right] \text{ and } 0 \leq t \leq T$$

representing the wealth of an investor at time t when holding q contracts of $\bar{C}_t$ and trading in S_t. Our goal is now to find the optimal strategy of investing into S_t, given an agent holds q units of $h(Y_T)$. Due to the fact that the uncertainty of S_t and Y_t have the same source, our conjecture for the optimal strategy is that it is optimal to hedge away all the risk arising from $h(Y_T)$ by a Delta hedge in S_t. Motivated by this, we introduce the following notation:

$$\bar{C}_t^Y := \frac{\partial}{\partial Y}\mathbb{E}^{\bar{\mathbb{Q}}}\left[h(Y_T)|\mathcal{F}_t\right], \bar{C}_t^{YY} := \frac{\partial}{\partial Y}\bar{C}_t^Y. \tag{3.2.7}$$

By the famous option pricing PDE derived by Black-Scholes-Merton, we get that $\bar{C}_t$ satisfies

$$\frac{\partial}{\partial t}\bar{C}_t + \bar{C}_t^Y Y_t r + \frac{1}{2}\bar{C}_t^{YY} Y_t^2 \eta^2 - r\bar{C}_t = 0,$$

which reduces to, as we assume $r = 0$,

$$\frac{\partial}{\partial t}\bar{C}_t + \frac{1}{2}\bar{C}_t^{YY} Y_t^2 \eta^2 = 0. \tag{3.2.8}$$

It follows, using the relationship in (3.2.5), Itô's formula, the fact that $B_t \overset{\text{law}}{=} Z_t$ and above reduced PDE, that

$$\begin{aligned} d\bar{X}_t &= dX_t + qd\bar{C}_t = \tilde{\pi}_t(\mu dt + \sigma B_t) + q\bar{C}_t^Y dY_t \\ &= \tilde{\pi}_t(\mu dt + \sigma dB_t) + q\bar{C}_t^Y (Y_t \nu dt + Y_t \eta dZ_t) \\ &= \left(\tilde{\pi}_t + \frac{q\bar{C}_t^Y Y_t \eta}{\sigma}\right)\sigma dB_t + \left(\tilde{\pi}_t + \frac{q\bar{C}_t^Y Y_t \eta}{\sigma}\right)\mu dt \\ &= \tilde{\Pi}_t \left(\sigma dB_t + \mu dt\right) \end{aligned}$$

for $\tilde{\Pi}_t := \left(\tilde{\pi}_t + \frac{q\bar{C}_t^Y Y_t \eta}{\sigma}\right)$.[10] This means, we can interpret this model as a classical Merton wealth problem as introduced in Section 3.1 with a modified strategy $\tilde{\Pi}_t$ instead of $\tilde{\pi}_t$ and get directly the optimal strategy, value function and average utility indifference price - we do not have to switch to the dual problem and minimize there over $y \geq 0$ and hoping to find a solution with zero duality gap.
Nevertheless, we will present the dual approach in the case of power law utility.

3.2.1.1 Complete Basis Risk Model under Power Law Utility

From Section 3.1 we have that the optimal amount $\tilde{\Pi}_t^\star$ of cash invested in the risky assets (here: S_t and $q\bar{C}_t$) for the power law utility $U_R(x) = \frac{x^{1-R}}{1-R}$ is given by

$$\tilde{\Pi}_t^\star = \frac{\mu}{\sigma^2 R}\bar{X}_t = \frac{\mu}{\sigma^2 R}\left(X_t + q\bar{C}_t\right),$$

meaning that the optimal quantity $\tilde{\pi}_t^\star$ of cash invested in S_t is given by [11]

$$\tilde{\pi}_t^\star(X_t, q; h) = \tilde{\Pi}_t^\star - \frac{q\bar{C}_t^Y Y_t \eta}{\sigma} = \frac{\mu}{\sigma^2 R}(X_t + q\bar{C}_t) - \frac{q\bar{C}_t^Y Y_t \eta}{\sigma}.$$

This leads to the value function

$$u_{U_R}(X_t, q; h) = \mathbb{E}^{\mathbb{P}}[U(\bar{X}_T)] = \frac{X_t^{1-R}}{1-R}\exp\left(\frac{1}{2}\frac{\mu^2}{\sigma^2}\frac{1-R}{R}(T-t)\right)\left(1 + \frac{q\bar{C}_t}{X_t}\right)^{1-R}, \tag{3.2.9}$$

based on the results seen in Section 3.1.
The interpretation of this result is as follows: The optimal quantity $\tilde{\pi}_t^\star$ of cash invested in the risky and traded assets S_t resp. $\bar{C}_t$ is given by a constant fraction of current wealth $\frac{\mu}{\sigma^2 R}\bar{X}_t$ plus (resp. minus) an additional Delta hedging term, where $\bar{X}_t = X_t + q\bar{C}_t$ is the agent's wealth at time t. In other words, the agent hedges all the risk away by a Delta hedge and then puts a constant fraction of her wealth into the risky asset. Based on the claim $h(Y_T)$, the Delta hedge can in- resp. decrease the overall money put into S_t.

We now want to verify these results using the dual approach:

[10] At this point, we have to pay attention that the admissibility property is not violated. As our results in this section are of general nature (without Assumption 2), we consider the four standard options types. For Put options (which have bounded payoff profile), admissibility is guaranteed. For long a Call option, admissibility is also ensured as $\bar{C}_t^Y \geq 0$. For short a Call option and other exotic claims, we must use a stopping argument that guarantees that $\tilde{\Pi}_t$ remains uniformly bounded from below.

[11] When applying the modified strategy $\tilde{\Pi}_t$ to the power utility case, we have to ensure that wealth remains positive. As in above footnote, we must use a stopping argument in the case where $q < 0$ or $\bar{C}_t^Y \leq 0$.

Our initial goal was to find the optimal strategy in the primal problem with claim $qh(Y_T)$

$$u_{U_R}(x,q;h) = \sup_{\tilde{\pi}\in H} \mathbb{E}^{\mathbb{P}}\left[U_R\left(x + \int_0^T \tilde{\pi}_t \frac{dS_t}{S_t} + qh(Y_T)\right)\right].$$

Turning our attention to the dual problem, we have seen that the convex conjugate is given by $V_R(y) = \frac{R}{1-R} y^{\frac{R-1}{R}}$. Hence it is easily seen that

$$\mathbb{E}^{\mathbb{P}}\left[V_R\left(y\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}\right)\right] = \frac{R}{1-R} y^{\frac{R-1}{R}} \exp\left(\frac{1}{2}\frac{\mu^2}{\sigma^2 R}\frac{1-R}{R}T\right).$$

Therefore the following has to be minimized over $y \geq 0$

$$\frac{R}{1-R} y^{\frac{R-1}{R}} \exp\left(\frac{1}{2}\frac{\mu^2}{\sigma^2 R}\frac{1-R}{R}T\right) + y(x + q\mathbb{E}^{\bar{\mathbb{Q}}}[h(Y_T)]),$$

which then gives that

$$u_{U_R}(x,q;h) \leq \frac{1}{1-R}\exp\left(\frac{1}{2}\frac{\mu^2}{\sigma^2 R}(1-R)T\right)(x + q\mathbb{E}^{\bar{\mathbb{Q}}}[h(Y_T)])^{1-R}. \tag{3.2.10}$$

Hence, as seen in (2.1.1), the dual approach provides us with an upper bound to the value function. However this inequality is in fact an equality as one easily can show that $\bar{\mathbb{Q}}$ is a martingale measure having finite (generalized) relative entropy with respect to $\mathbb{P}$, hence we have satisfied the criterion of equivalence between (Primal Problem) and (Dual Problem) is satisfied.
An alternative and more rigorous argument is that the right-hand side of (3.2.10) can be seen as

$$u_{U_R}(x,q;0)\left(1 + \frac{q}{x}\mathbb{E}^{\bar{\mathbb{Q}}}[h(Y_T)]\right)^{1-R}.$$

Using the fact that

$$u_{U_R}(x,q;h) \geq u_{U_R}(x,q;0),$$

gives us the desired equality.

3.2.1.2 Complete Basis Risk Model under Exponential Utility

Similar results can be obtained for the exponential utility function $U_\alpha(x) = -\frac{1}{\alpha}e^{-\alpha x}$. The optimal quantity $\tilde{\pi}_t^\star$ of cash invested in S_t is given by

$$\pi_t^\star(X_t,q;h) = \frac{\mu}{\sigma^2\alpha} - \frac{\eta q}{\sigma}\bar{C}_t^Y Y_t.$$

The value function is given by, using again the results from Section 3.1

$$u_{U_\alpha}(X_t,q;h) = -\frac{1}{\alpha}\exp\left(-\alpha(X_t + q\bar{C}_t)\right)\exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t)\right).$$

The interpretation stays the same: The additional term is due to the fact that the agents hedges all the risk away arising from price movements in the underlying.
Also in this case, the dual approach would directly give us the desired solution with zero duality gap as $\mathbb{Q}_{\min} \in \tilde{\mathcal{M}}^n$.

3.2.1.3 Average Utility Indifference Price in Complete Basis Risk Model

By the classical utility indifference criterion

$$u_U(x,q;0) = u_U(x - qp_U(x,q;h), q; h),$$

we get the average utility indifference price $p_U(x,q;h)$ in an explicit form:

$$\frac{x^{1-R}}{1-R}\exp\left(\frac{1-R}{R}\frac{\mu^2}{2\sigma^2}(T-t)\right) \stackrel{!}{=} \frac{(x-qp_U)^{1-R}}{1-R}\exp\left(\frac{1-R}{R}\frac{\mu^2}{2\sigma^2}(T-t)\right)\left(1+\frac{q\bar{C}_t}{x-qp_U}\right)^{1-R},$$

which clearly reduces to

$$p_{U_R}(x,q;h) = \bar{C}_t = \mathbb{E}^{\bar{\mathbb{Q}}}\left[h(Y_T)|\mathcal{F}_t\right].$$

Similar calculations show also for the exponential utility that

$$p_{U_\alpha}(x,q;h) = \mathbb{E}^{\bar{\mathbb{Q}}}\left[h(Y_T)|\mathcal{F}_t\right].$$

In this case, average utility indifference prices coincide with the Black-Scholes prices.

Moreover, in the case of exponential utility, we can even represent the price as follows

$$p_{U_\alpha}(x,q;h) = \frac{1}{\alpha q}\log\left(\frac{u_{U_\alpha}(0,q;0)}{u_{U_\alpha}(0,q;h)}\right). \tag{3.2.11}$$

In fact, this is exactly what we should have expected. As we are in a complete market, there is a unique martingale measure $\bar{\mathbb{Q}}$ which gives us conditions on the drift and volatility of the two considered assets S_t and Y_t, namely that their Sharpe ratios have to coincide (and vanish under $\bar{\mathbb{Q}}$), which then gives an arbitrage-free model. The arbitrage-free price is simply derived by computing the conditional expectation of the claim under the respective unique martingale measure.

This is exactly what above calculations show: the utility indifference price $p_U(x,q;h)$ is purely independent of the individual risk aversion and is given by the arbitrage-free Black-Scholes price. In other words, the price for which an investor is indifferent between holding q contracts of $h(Y_T)$ and trading in S_t and only trading in S_t is equal to the fair price for which there is no arbitrage in the market. Note that the price is also independent of q, hence the price for q units of $h(Y_T)$ is linear in q, which won't be the case anymore under incompleteness as there will always be some risk left and the individual aversion towards risk comes into play. It can therefore be concluded that the power of no arbitrage is stronger and purely determines market prices than the power of the individual risk aversion, which has no impact at all.

Lastly, we want to emphasize that in this whole study presented up to now, Assumption 1 does not have to be satisfied due to the utility independence of our results. Hence above formula is also valid for e.g. a 'short Call' position among others.

But in contrast to that, when we are dealing with an incomplete market framework, this relationship won't exist anymore and the individual utility will have a more significant effect on the results.

3.2.2 Incomplete Basis Risk Model

In what follows, we present the case where $|\varrho| < 1$. It turns out, that not all risk can be described by one Brownian motion, as there is a second one not perfectly correlated to the first. Hence, we are in a setting with two, not identical sources of risk. Clearly, any position in $h(Y_T)$ cannot be replicated by just

using S_t, thus the market is incomplete and some unhedgeable risk is always left over. Therefore, the individual aversion towards risk will play a crucial role and will have a significant impact on the results. On the other hand, Assumption 1 will come into play and 'short Call' options are not anymore possible to be treated.
We study two cases separately, beginning with the incomplete market model with power law utility.

3.2.2.1 Incomplete Basis Risk Model with Power Law Utility

We will now state the main theorem of this chapter and present a rigorous proof. It gives us the optimal strategy of investing into S_t and from this the value function which then can be used to derive average utility indifference prices.

Theorem 3.2.1. ([Hen02, Theorem 4.1])

1. *Define* $C_t := \mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]$ *and* $C_t^Y := \frac{\partial}{\partial Y}\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]$. *Then for h and q satisfying Assumption 1, the optimal strategy is given by*

$$\tilde{\pi}_t^{\star}(X_t, q; h) = \tilde{\pi}_t^1(X_t, q; h) + o(q) \text{ for } q \to 0,$$

where

$$\tilde{\pi}_t^1(X_t, q; h) := \frac{\mu}{\sigma^2 R}(X_t + qC_t) - \varrho\frac{\eta}{\sigma}qC_t^Y Y_t.$$

2. *Using* $\tilde{\pi}_t^1$, *we define*

$$u_{U_R}^1(X_t, q; h) := \frac{X_t^{1-R}}{1-R} e^{\frac{1-R}{R}\frac{\mu^2}{2\sigma^2}(T-t)} \left(1 + q\frac{C_t}{X_t} - \frac{q^2}{2}R\eta^2(1-\varrho^2)\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_t^T \frac{Y_u^2(C_u^Y)^2}{(X_u^0)^2}du\right]\right)^{1-R},$$

where

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} := \mathcal{E}\left(\frac{\mu(1-R)}{\sigma R} \cdot B\right)_T,$$

and X_u^0 *is the optimal wealth derived by following the optimal strategy* $\tilde{\pi}_t^0 := \frac{\mu}{\sigma^2 R}X_t^0$. *Then for* q, h *satisfying Assumption 1, the value function is given by*

$$u_{U_R}(X_t, q; h) = u_{U_R}^1(X_t, q; h) + o(q^2) \text{ for } q \to 0.$$

Remark 3.2.1.

- Theorem 3.2.1 shows that even if markets are not complete and agents are exposed to some unhedgeable remaining risk, the agent still invests a constant fraction of his current wealth into S_t and additionally hedges away the risk arising from B_t by a (proxy) Delta hedge. The magnitude of this Delta hedge heavily depends on the correlation and the larger the correlation in magnitude is, the higher weight the Delta hedge term carries in the overall strategy. Intuitively, this is clear: The closer related resp. correlated the two assets S_t and Y_t are, the more profitable it is to go into a certain position of S_t for hedging purposes.

- We will later establish and investigate using some concrete examples the impact of the second order term and the error an investor would made if she uses naively the hedging strategy from the complete case in an incomplete market scenario. We refer to Section 3.3.

- Concerning the second statement of above theorem, we see that $\mathbb{Q}_{\min}$ was a good guess for a minimizing probability measure as it (or a slightly adjusted version of it) appears in various arguments. For example, the measure $\hat{\mathbb{P}}$ is obtained from $\mathbb{Q}_{\min}$ by a numeraire change (see Chapter 6 for more details).
  Moreover, in the proof we will derive lower and upper bounds in terms of $u^1_{U_R}(x,q;h)$. For the latter we switch to the dual problem and it turns out that also there a modified version of the minimal martingale measure will help us out.

- We also want to point out the appearance of ϱ^2 and by this the symmetry in the value function seen as a function of ϱ. This is due to the different hedging positions an investor can take in, meaning that if we consider two scenarios with risky assets S_t, Y_t having correlations ϱ_1 resp. ϱ_2 with $1 > \varrho_1 = -\varrho_2 > 0$, then this yields in the same value function (i.e. the same maximal expected utility at time T) and the approximative hedge (i.e. the positions in S_t) differs then just by the sign, i.e. by a long resp. short position, but gives the same protection against risk.
  For perfectly correlated assets S_t and Y_t, the value function remains the same as in the complete market case. In contrast to this, for non-correlated assets (i.e. $\varrho = 0$), an agent would not take any money aside for hedging, as hedging is purely useless, and this yields in a (maximal) deduction in the value function as there is unhedgeable risk left over.

- Lastly, we want to emphasize that the expression of $u^1_{U_R}(x,q;h)$ and the value function from the complete market model agree up to order q. The additional second-order term

$$\frac{q^2}{2} R\eta^2(1-\varrho^2)\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_t^T \frac{Y_u^2 (C_u^Y)^2}{(X_u^0)^2} du\right],$$

  which is always nonnegative, can be seen as a deduction from the initial value function from the complete case due to the presence of unhedgeable risk. It is given by a constant times the over time and over paths averaged square of a (scaled) Delta hedge term. We do not directly see the exact intuition in this setting for this term. That is why we will consider a general semimartingale model in the small claim limit (see Chapter 6), and surprisingly, this term will appear there as well in a more intuitive setting.

Let's turn our attention to the proof of this powerful theorem.

Proof. ([Hen02, Theorem 4.1]) The idea is as follows: We will show that the strategy $\tilde{\pi}^1_t$ is optimal by deriving upper and lower bounds for the supremum of expected utility agreeing up to order q^2 with $u^1_{U_R}(x,q;h)$.
For the lower bound, we consider a cleverly chosen strategy (with a localizing argument that guarantees that wealth remains positive) and derive for the value function a lower bound in terms of $u^1_{U_R}(x,q;h)$. To find an upper bound in terms of $u^1_{U_R}(x,q;h)$, we consider the dual problem and a slightly adjusted version of $\mathbb{Q}_{\min}$. [HH04]
It will be crucial in our study to have Assumption 1 as we will need that $qh(Y_T) \geq 0$. This is satisfied under Assumption 1 (ii). For (i), we have to distinguish two cases. If $q > 0$, then $qh(Y_T) \geq 0$. Else, write $-|q|h = -|q|B + |q|(B-h)$. Thus the payoff at time T can be split into a positive part minus a constant. Lastly, we note that adding constants to claims does not have any impact on the second order terms.
For simplicity, we set $t = 0$. Of course, everything could be proven in the case $0 < t < T$, we just have to replace expectations by conditional expectations.
Now we can turn our attention to the value function and derive a lower bound in terms of $u^1_{U_R}(x,q;h)$.

Lower bound

In a first step, we consider the case where we have no claim $h(Y_T)$. This case was established previously. Denote by X_t^0 and $\tilde{H}_t^0 = \frac{\tilde{\pi}_t^0}{X_t^0}$ the optimal wealth resp. relative amount of cash invested in S_t. Then we have seen that $dX_t^0 = \tilde{\pi}_t^0 \frac{dS_t}{S_t}$ with $\tilde{\pi}_t^0 = \frac{\mu}{\sigma^2 R} X_t^0$ is given by

$$X_t^0 = x_0 \exp\left(\frac{\mu^2}{\sigma^2 R} t - \frac{\mu^2}{2\sigma^2 R^2} t + \frac{\mu}{\sigma R} B_t\right).$$

We note here that under the minimal martingale measure $\mathbb{Q}_{\min}$, we have that X_t^0 is a martingale. Indeed

$$\begin{aligned} X_t^0 &= x_0 \exp\left(\frac{\mu^2}{\sigma^2 R} t - \frac{\mu^2}{2\sigma^2 R^2} t + \frac{\mu}{\sigma R}\left(\tilde{B}_t - \frac{\mu}{\sigma} t\right)\right) \\ &= x_0 \exp\left(\frac{\mu}{\sigma R} B_t^{\min} - \frac{1}{2}\frac{\mu^2}{\sigma^2 R^2} t\right) = \mathcal{E}\left(\frac{\mu}{\sigma R} \cdot B^{\min}\right)_t, \end{aligned}$$

where $B_t^{\min} := B_t + \frac{\mu}{\sigma} t$ is a $\mathbb{Q}_{\min}$-Brownian motion.
We use now the aforementioned localizing argument to guarantee that wealth remains positive as we work with power law utilities which are only supporting positive wealths. For a fixed $K > 0$, define

$$\tau_K = \inf\left\{u \geq 0 : \int_0^u \frac{1}{X_t^0}\left(-\frac{\mu}{\sigma^2 R} C_t + \frac{\eta\varrho}{\sigma} Y_t C_t^Y\right)\left(\frac{dS_t}{S_t} - \frac{\mu dt}{R}\right) = K\right\}.$$

Suppose that $K > 0$ and $q < \frac{1}{2}K^{-1}$. Consider the wealth process $X_t^{1,K}$ generated from an initial wealth $x_0 > 0$ using the stopped strategy

$$\tilde{\pi}_t^{1,K} := \frac{\mu}{\sigma^2 R}\left(X_t^{1,K} + qC_t \mathbf{1}_{t<\tau_K}\right) - \frac{\eta\varrho}{\sigma} q Y_t C_t^Y \mathbf{1}_{t<\tau_K}.$$

Then

$$dX_t^{1,K} = \tilde{\pi}_t^{1,K} \frac{dS_t}{S_t} = \frac{\mu}{\sigma^2 R} X_t^{1,K} \frac{dS_t}{S_t} + \frac{\mu}{\sigma^2 R} q C_t \mathbf{1}_{t<\tau_K} \frac{dS_t}{S_t} - \frac{\eta\varrho}{\sigma} q Y_t C_t^Y \mathbf{1}_{t<\tau_K} \frac{dS_t}{S_t},$$

which then gives that $X_t^{1,K}$ can be written in a generalized stochastic exponential form[12]

(3.2.12) $$X_t^{1,K} = \int_0^t X_s^{1,K} dZ_s + H_t,$$

for $dZ_s := \left(\frac{\mu}{\sigma^2 R} \cdot \frac{dS}{S}\right)_s$ and $dH_t := q\mathbf{1}_{t<\tau_K}\left(\frac{\mu}{\sigma^2 R} C_t - \frac{\eta\varrho}{\sigma} Y_t C_t^Y\right)\frac{dS_t}{S_t}$. By [Pro04, Theorem V.52], we have that $X_t^{1,K}$ is explicitly given by

(3.2.13) $$X_t^{1,K} = X_t^0\left(1 + q\int_0^{t\wedge\tau_K} \frac{1}{X_u^0}\left[\frac{\mu}{\sigma^2 R} C_u - \frac{\eta\varrho}{\sigma} Y_u C_u^Y\right]\left[\frac{dS_u}{S_u} - \frac{\mu}{R} du\right]\right).$$

We note that on the event $\{\tau_K < T\}$, we have that

$$X_T^{1,K} = X_T^0(1 - qK)$$

[12] Note that the well-known stochastic exponential $X_t = \mathcal{E}(Z)_t$ is defined trough $X_t = 1 + \int_0^t X_s dZ_s$ for a continuous semimartingale Z_t. Here we have also an exogenous driving term H_t which make the calculations less straightforward. The solution can be derived by variation of constants and is given by $X_t = \mathcal{E}_H(Z)_t = \mathcal{E}(Z)\left(H_0 + \int_{0+}^t \mathcal{E}(Z)_s^{-1} d(H_s - \langle H, Z\rangle_s)\right)$, see [Pro04, Theorem V.52].

and more generally

$$X_T^{1,K} \geq X_T^0(1-qK),$$

meaning that the wealth process is bounded from below and positive as $q < \frac{1}{2}K^{-1}$. Moreover, we have that $X_t^{1,K}$ is a $\mathbb{Q}_{\min}$-martingale. Indeed we have from [Pro04, Theorem V.6, Theorem V.7] that $X_t^{1,K}$ is a semimartingale. By applying Itô's formula, one easily sees that the drift of $X_t^{1,K}$ under $\mathbb{Q}_{\min}$ vanishes. Then by the uniform boundedness from below, we then have that $X_t^{1,K}$ is a supermartingale. Lastly, the martingale property follows by noticing that the term $(\frac{\mu}{\sigma R}C_t - \eta\varrho Y_t C_t^Y)$ affects the martingale property. Under the assumption of at least polynomial growth of our claim h, this term is square-integrable which gives us that the stochastic integral is indeed a martingale and by this $X_t^{1,K}$ as well.
Consider $Z_T^{q,K} := X_t^{1,K} + qC_t$, which gives on $\{\tau_K \geq t\}$ using the PDE satisfied by C_t

$$\begin{aligned}
dZ_t^{q,K} &= \tilde{\pi}_t^{1,K}\frac{dS_t}{S_t} + q\left(\partial_t C_t dt + C_t^Y dY_t + \frac{1}{2}C_t^{YY}Y_t^2\eta^2 dt\right) \\
&= \tilde{\pi}_t^{1,K}\frac{dS_t}{S_t} + qC_t^Y\left(Y_t\nu dt + Y_t\eta\left(\varrho dB_t + \sqrt{1-\varrho^2}dW_t\right)\right) \\
&= \frac{\mu}{\sigma^2 R}Z_t^{q,K}\frac{dS_t}{S_t} + qY_tC_t^Y\left(\nu dt - \frac{\eta\varrho}{\sigma}\mu dt\right) + qY_tC_t^Y\eta\sqrt{1-\varrho^2}dW_t \\
&= \frac{\mu}{\sigma^2 R}Z_t^{q,K}\frac{dS_t}{S_t} + qY_tC_t^Y\eta\sqrt{1-\varrho^2}dW_t,
\end{aligned}$$

where we used in the last equality the fact that $X_t^{1,K}$ as well as C_t are $\mathbb{Q}_{\min}$-martingales which yields that the drift of $Z_t^{q,K}$ has to vanish under $\mathbb{Q}_{\min}$[13]. This implies that we have that $Z_t^{q,K}$ can be represented in the form of (3.2.12) with $dZ_s := \left(\frac{\mu}{\sigma^2 R}\cdot\frac{dS}{S}\right)_s$ and $dH_t := qY_tC_t^Y\eta\sqrt{1-\varrho^2}dW_t$ and hence by [Pro04, Theorem V.52], we have that on $\{\tau_K \geq t\}$

$$Z_t^{q,K} = X_t^0\left(1 + q\left(\frac{\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]}{x_0} + \int_0^t \frac{Y_u C_u^Y}{X_u^0}\eta\sqrt{1-\varrho^2}dW_u\right)\right).$$

We record that

$$Z_T^{q,K} = X_T^{1,K} + qh(Y_T) \geq X_T^0(1-qK) + qh(Y_T) \geq X_T^0(1-qK),$$

which is bounded from below. Here, we used the Assumption of $qh(Y_T) \geq 0$. Let's turn our attention to the utility function $U_R(x)$:
Using Taylor expansion for $U_R(x)$ around X_T^0 gives, for $0 \leq \xi \leq 1$

$$U_R(Z_T^{q,K}) = U_R(X_T^0) + (Z_T^{q,K} - X_T^0)U_R'(X_T^0) + \frac{1}{2}(Z_T^{q,K} - X_T^0)^2 U_R''(X_T^0 + \xi(Z_T^{q,K} - X_T^0)).$$

Now, consider the $\mathbb{P}$-expectation of above term. The first term gives $\mathbb{E}^{\mathbb{P}}[U_R(X_T^0)] = u_{U_R}(x_0, q; 0)$, as the strategy $\tilde{\pi}_t^0$ was chosen optimal. Moreover, note that

$$U_R'(X_T^0) = \frac{1}{x_0^R}\exp\left(\frac{\mu^2}{2\sigma^2}\frac{1-R}{R}T\right)\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}}.$$

[13] To be more precise, we know that the $\mathbb{Q}_{\min}$-dynamics of $\frac{dS_t}{S_t}$ are given by $\sigma d\tilde{B}_t$ and that $\tilde{W}_t = W_t$ is a $\mathbb{Q}_{\min}$-Brownian motion. Hence the first and third term together build a $\mathbb{Q}_{\min}$-martingale, hence the middle term, the drift, has to vanish to guarantee that everything together remains a martingale.

It follows for the second term that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\left(Z_T^{q,K}-X_T^0\right)U_R'(X_T^0)\right] &= \mathbb{E}^{\mathbb{P}}\left[X_T^{1,K}U_R'(X_T^0)\right]+\mathbb{E}^{\mathbb{P}}\left[qC_TU_R'(X_T^0)\right]-\mathbb{E}^{\mathbb{P}}\left[X_T^0U_R'(X_T^0)\right]\\ &= x_0^{-R}\exp\left(\frac{1-R}{R}\frac{\mu^2}{2\sigma^2}T\right)\mathbb{E}^{\mathbb{Q}_{\min}}\left[qh(Y_T)\right],\end{aligned}$$

where we have used in the third equation that X_t^0 and $X_t^{1,K}$ are martingales under $\mathbb{Q}_{\min}$ and hence the first and third term vanish. Indeed, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[X_T^{1,K}U_R'(X_T^0)\right] &= \mathbb{E}^{\mathbb{P}}\left[X_T^{1,K}x_0^{-R}\exp\left(\frac{\mu^2}{2\sigma^2}\frac{1-R}{R}T\right)\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}}\right]\\ &= x_0^{-R}\exp\left(\frac{\mu^2}{2\sigma^2}\frac{1-R}{R}T\right)\mathbb{E}^{\mathbb{Q}_{\min}}\left[X_T^{1,K}\right]=0,\end{aligned}$$

and

$$\mathbb{E}^{\mathbb{P}}\left[X_T^0U_R'(X_T^0)\right]=x_0^{-R}\exp\left(\frac{\mu^2}{2\sigma^2}\frac{1-R}{R}T\right)\mathbb{E}^{\mathbb{Q}_{\min}}\left[X_T^0\right]=0.$$

For the third term in Taylor's expansion, we have that $X_T^0+\xi(Z_T^{q,K}-X_T^0)\geq X_T^0(1-qK)$ by setting $\xi=1$. It follows that on $\{\tau_K\geq T\}$, we have that $(Z_T^{q,K}-X_T^0)=qX_T^0\left(\frac{\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]}{x_0}+\int_0^T\frac{Y_uC_u^Y}{X_u^0}\eta\sqrt{1-\varrho^2}dW_u\right)$ and that on $\{\tau_K<T\}$, we have that $(Z_T^{q,K}-X_T^0)=q(C_T-X_T^0K)$. Then, as $U_R''(x)$ is increasing,

$$\begin{aligned}&q^{-2}(Z_T^{q,K}-X_T^0)^2U_R''(X_T^0+\xi(Z_T^{1,K}-X_T^0))\\ &\geq (X_T^0)^2\left(\frac{\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]}{x_0}+\int_0^T\frac{Y_tC_t^Y}{X_t^0}\eta\sqrt{1-\varrho^2}dW_t\right)^2U_R''(X_T^0(1-qK))\mathbf{1}_{\tau_K\geq T}\\ &+\left(h(Y_T)-X_T^0K\right)^2U_R''(X_T^0(1-qK))\mathbf{1}_{\tau_K<T}.\end{aligned}$$

Taking expectations and the limit as $q\to 0$ and rearranging the terms in Taylor's expansion yields

$$\begin{aligned}&\lim_{q\to 0}q^{-2}\left(\mathbb{E}^{\mathbb{P}}[U_R(Z_T^{q,K})]-\mathbb{E}^{\mathbb{P}}[U_R(X_T^0)]-q\mathbb{E}^{\mathbb{P}}[h(Y_T)U_R'(X_T^0)]\right)\\ &=\frac{1}{2}\lim_{q\to 0}q^{-2}\mathbb{E}^{\mathbb{P}}\left[(Z_T^{q,K}-X_T^0)^2U_R''(X_T^0+\xi(Z_T^{1,K}-X_T^0))\right]\\ &\geq\frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[(X_T^0)^2\left(\frac{\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]}{x_0}+\int_0^T\frac{Y_tC_t^Y}{X_t^0}\eta\sqrt{1-\varrho^2}dW_t\right)^2U_R''(X_T^0)\mathbf{1}_{\tau_K\geq T}\right]\\ &+\frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\left(h(Y_T)-X_T^0K\right)^2U_R''(X_T^0)\mathbf{1}_{\tau_K<T}\right],\end{aligned}\tag{3.2.14}$$

where we used the dominated convergence theorem to change the order of the limit and the integral. If we let $K\to\infty$, the lower bound in (3.2.14) becomes

$$\frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[(X_T^0)^2\left(\frac{\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]}{x_0}+\int_0^T\frac{Y_tC_t^Y}{X_t^0}\eta\sqrt{1-\varrho^2}dW_t\right)^2U_R''(X_T^0)\right].\tag{3.2.15}$$

We define a new probability measure $\hat{\mathbb{P}}$ by $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}:=\mathcal{E}\left(\frac{\mu(1-R)}{\sigma R}\cdot B\right)_T$, and note that $\hat{B}_t:=B_t-\frac{\mu(1-R)}{\sigma R}t$ as well as $\hat{W}_t:=W_t$ are Brownian motions under this new measure.

Then (3.2.15) becomes, using the known form of $U''(x) = -Rx^{-R-1}$

$$-\frac{R}{2}\mathbb{E}^{\mathbb{P}}\left[(X_T^0)^{1-R}\left(\frac{\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]}{x_0}+\int_0^T\frac{Y_tC_t^Y}{X_t^0}\eta\sqrt{1-\varrho^2}dW_t\right)^2\right]$$

$$= -\frac{R}{2}x_0^{1-R}\exp\left(\frac{1-R}{R}\frac{\mu^2}{2\sigma^2}T\right)\mathbb{E}^{\hat{\mathbb{P}}}\left[\left(\frac{\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]}{x_0}+\int_0^T\frac{Y_tC_t^Y}{X_t^0}\eta\sqrt{1-\varrho^2}dW_t\right)^2\right]$$

$$= -\frac{R}{2}x_0^{1-R}\exp\left(\frac{1-R}{R}\frac{\mu^2}{2\sigma^2}T\right)\left[\left(\frac{\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]}{x_0}\right)^2+\eta^2(1-\varrho^2)\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^T\frac{(Y_tC_t^Y)^2}{(X_t^0)^2}dt\right]\right]. \tag{3.2.16}$$

Summarizing, we have shown that

$$\limsup_{K\to\infty}\lim_{q\to 0}\left(\mathbb{E}^{\mathbb{P}}[U_R(Z_T^{q,K})]-\mathbb{E}^{\mathbb{P}}[U_R(X_T^0)]-q\mathbb{E}^{\mathbb{P}}[h(Y_T)U_R'(X_T^0)]\right)\geq (3.2.16)$$

and finally we have, using $(1+z)^\alpha \leq 1+\alpha z+\frac{\alpha(\alpha-1)}{2}z^2$ for $z\geq 0$ and $\alpha\leq 1$

$$\begin{aligned}u_{U_R}(x_0,q;h) &\geq \mathbb{E}^{\mathbb{P}}[U_R(Z_T^{q,K})]\\ &\geq u_{U_R}(x_0,q;0)+qx_0^{-R}\exp\left(\frac{1-R}{R}\frac{\mu^2}{2\sigma^2}T\right)\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]\\ &\quad -q^2\frac{R}{2}x_0^{1-R}\exp\left(\frac{1-R}{R}\frac{\mu^2}{2\sigma^2}T\right)\left[\left(\frac{\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]}{x_0}\right)^2+\eta^2(1-\varrho^2)\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^T\frac{(Y_tC_t^Y)^2}{(X_t^0)^2}dt\right]\right]\\ &\quad +o(q^2)\\ &\geq u^1_{U_R}(x_0,q;h)+o(q^2),\end{aligned}$$

which gives us the desired lower bound.

Upper bound

Based on (2.1.1), we will apply the dual approach for finding an upper bound on the value function $u_{U_R}(x,q;h)$ in terms of $u^1_{U_R}(x,q;h)$. Then we are going to choose the probability measure in a clever (closely related with $\mathbb{Q}_{\min}$) way to obtain a high order bound.
Our goal is to show that $u_{U_R}(x,q;h)\leq u^1_{U_R}(x,q;h)+\varepsilon q^2$ for any $\varepsilon>0$.
For this, we define

$$M_t := \eta\sqrt{1-\varrho^2}\int_0^t\frac{Y_uC_u^Y}{X_u^0}dW_u,\quad 0\leq t\leq T.$$

And for $K>0$ define

$$T_K := \inf\{t\geq 0 : |M_t|+\langle M\rangle_t = K\}.$$

Choose K large enough such that for some fixed $\varepsilon>0$

$$\mathbb{E}^{\hat{\mathbb{P}}}\left[\langle M\rangle_T-\langle M\rangle_{T_K}\right]<\varepsilon.$$

We introduce a new probability measure $\mathbb{Q}_K$ given by

$$\frac{d\mathbb{Q}_K}{d\mathbb{P}}=\exp\left(-\frac{\mu}{\sigma}B_T-\frac{\mu^2}{2\sigma^2}T\right)\exp\left(-RqM_{T_K}-\frac{1}{2}R^2q^2\langle M\rangle_{T_K}\right).$$

We remark at this point that $\mathbb{Q}_K$ is closely related to $\mathbb{Q}_{\min}$, that is, we have

$$\frac{d\mathbb{Q}_K}{d\mathbb{P}} = \frac{d\mathbb{Q}_{\min}}{d\mathbb{P}} \exp\left(-RqM_{T_K} - \frac{1}{2}R^2q^2\langle M\rangle_{T_K}\right).$$

Then we have for the convex conjugate $V_R(y)$ that the generalized entropy term is given by

$$\mathbb{E}^{\mathbb{P}}\left[V_R\left(y\frac{d\mathbb{Q}_K}{d\mathbb{P}}\right)\right] = \frac{R}{1-R}y^{\frac{R-1}{R}}A_K,$$

for A_K given by (recall the definition of $\hat{\mathbb{P}}$ in the proof of the lower bound)

$$\begin{aligned} A_K &= \exp\left(-\frac{\mu(R-1)}{\sigma R}B_T - \frac{\mu^2(R-1)}{2\sigma^2 R}T\right)\exp\left(-(R-1)qM_{T_K} - \frac{1}{2}R(R-1)q^2\langle M\rangle_{T_K}\right) \\ &= \exp\left(\frac{\mu^2}{2\sigma^2 R}\frac{1-R}{R}T\right)\mathbb{E}^{\hat{\mathbb{P}}}\left[\exp\left(\frac{1}{2}(1-R)q^2\langle M\rangle_{T_K}\right)\right]. \end{aligned}$$

By definition of T_K, we have that $\langle M\rangle_{T_K}$ is bounded by K, thus A_K can be expanded

$$\begin{aligned} A_K &= \exp\left(\frac{\mu^2}{2\sigma^2 R}\frac{1-R}{R}T\right)\left[1+\frac{1}{2}(1-R)q^2\mathbb{E}^{\hat{\mathbb{P}}}[\langle M\rangle_{T_K}] + O(q^4)\right] \\ &\le \exp\left(\frac{\mu^2}{2\sigma^2 R}\frac{1-R}{R}T\right)\left[1+\frac{1}{2}(1-R)q^2\left(\mathbb{E}^{\hat{\mathbb{P}}}[\langle M\rangle_{T_K}] - \varepsilon\mathbf{1}_{R>1}\right) + O(q^4)\right]. \end{aligned}$$

By the definition of $\mathbb{Q}_K$ and the explicit form of $\mathbb{Q}_{\min}$, we get

$$\mathbb{E}^{\mathbb{Q}_K}\left[h(Y_T)\right] = \mathbb{E}^{\mathbb{P}}\left[h(Y_T)\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}}\exp\left(-RqM_{T_K} - \frac{1}{2}R^2q^2\langle M\rangle_{T_K}\right)\right],$$

which can be written, using exponential expansion again, as

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_K}\left[h(Y_T)\right] &= \mathbb{E}^{\mathbb{Q}_{\min}}\left[h(Y_T)\left(1-qRM_{T_K}+o(q)\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}_{\min}}\left[h(Y_T)\right] - qR\mathbb{E}^{\mathbb{Q}_{\min}}\left[M_{T_K}h(Y_T)\right] + o(q). \end{aligned}$$

Let's turn our attention to M_t and C_t. By definition we have that C_t is a $\mathbb{Q}_{\min}$-martingale. It turns out, that M_t is a $\mathbb{Q}_{\min}$-martingale as well under the assumption of at most polynomial growth of the claim h and as W_t is also a $\mathbb{Q}_{\min}$-Brownian motion turning the stochastic integral into a (square-integrable) martingale. Therefore, using the stopping theorem (as T_K is finite a.s.), we conclude that the stopped process M_{T_K} is a martingale, too.

Thus Itô's formula resp. the formula of integration by parts for M_tC_t gives, using the martingale property and the definition of the quadratic covariation process $\langle\cdot,\cdot\rangle_t$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_{\min}}\left[M_{T_K}h(Y_T)\right] &= 0 + \mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^T M_{t\wedge T_K}dC_t\right] + \mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^T C_t dM_{t\wedge T_K}\right] + \mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^T d\langle M_{\cdot\wedge T_K}C_\cdot\rangle_t\right] \\ &= \mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^T d\langle M_{\cdot\wedge T_K}\rangle_t\right] = \eta^2(1-\varrho^2)\mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^{T_K}\frac{Y_t^2(C_t^Y)^2}{X_t^0}dt\right]. \end{aligned} \quad (3.2.17)$$

By the definition of X_T^0 and $\hat{\mathbb{P}}$, we get

$$\left(\frac{X_T^0}{x_0}\right)^{1-R} = \exp\left(\frac{\mu(1-R)}{\sigma R}B_T + \frac{\mu^2(1-R)}{\sigma^2 R}T - \frac{1}{2}\frac{\mu^2(1-R)}{\sigma^2 R^2}T\right) = \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\exp\left(\frac{\mu^2}{2\sigma^2}\frac{1-R}{R}T\right).$$

In the same way, we obtain

$$\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}} = \left(\frac{X_T^0}{x_0}\right)^{-R}\exp\left(\frac{\mu^2}{2\sigma^2}\frac{R-1}{R}T\right).$$

As mentioned earlier, we record that $\hat{\mathbb{P}}$ and $\mathbb{Q}_{\min}$ by a numeraire change

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{Q}_{\min}} = \frac{X_T^0}{x_0}. \tag{3.2.18}$$

We can use those representations to conclude that

$$\begin{aligned}\left(\mathbb{E}^{\hat{\mathbb{P}}}[\langle M\rangle_{T_K}] =\right)\mathbb{E}^{\hat{\mathbb{P}}}\left[M_{T_K}^2\right] &= \mathbb{E}^{\mathbb{P}}\left[\frac{M_{T_K}^2}{\exp\left(\frac{\mu^2}{2\sigma^2}\frac{R-1}{R}T\right)}\left(\frac{X_T^0}{x_0}\right)^{1-R}\right] \\ &= \frac{1}{x_0}\mathbb{E}^{\mathbb{Q}_{\min}}\left[M_{T_K}^2 X_T^0\right] \\ &= \frac{1}{x_0}\mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^{T_K} X_u^0 d\langle M^2\rangle_u\right] \\ &= \frac{\eta^2(1-\varrho^2)}{x_0}\mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^{T_K}\frac{Y_t^2(C_t^Y)^2}{X_t^0}dt\right],\end{aligned} \tag{3.2.19}$$

using again Itô's integration by parts formula as X_T^0 and M_t are $\mathbb{Q}_{\min}$-martingales.
Combining (3.2.19) and (3.2.17), we get that

$$\mathbb{E}^{\mathbb{Q}_{\min}}[M_{T_K}h(Y_T)] = x_0\mathbb{E}^{\hat{\mathbb{P}}}[M_{T_K}^2] = x_0\mathbb{E}^{\hat{\mathbb{P}}}[\langle M\rangle_{T_K}] < K, \tag{3.2.20}$$

as $\langle M\rangle_t$ is the unique right-continuous and increasing process such that $M_t^2 - \langle M\rangle_t$ is a martingale for a (square-integrable) martingale M_t. From this we can conclude that, using exponential expansion in the definition of $\mathbb{Q}_K$

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_K}[h(Y_T)] &= \mathbb{E}^{\mathbb{Q}_{\min}}\left[h(Y_T)\right] + qR\mathbb{E}^{\mathbb{Q}_{\min}}\left[M_{T_K}h(S_T)\right] + o(q) \\ &= \mathbb{E}^{\mathbb{Q}_{\min}}\left[h(Y_T)\right] - qRx_0\mathbb{E}^{\hat{\mathbb{P}}}\left[\langle M\rangle_{T_K}\right] + o(q).\end{aligned}$$

Finally, as we already have seen previously

$$\begin{aligned}u_{U_R}(x,q;h) &\leq \inf_{y\geq 0}\left\{\frac{R}{1-R}y^{\frac{R-1}{R}}A_K + yx + qy\mathbb{E}^{\mathbb{Q}_K}[h(Y_T)]\right\} \\ &= \frac{1}{1-R}(A_K)^R(x + q\mathbb{E}^{\mathbb{Q}_K}[h(Y_T)])^{1-R}\end{aligned}$$

and putting pieces together yields

$$\begin{aligned} u_{U_R}(x,q;h) &\leq \inf_{y\geq 0}\left\{\mathbb{E}^{\mathbb{P}}\left[V\left(y\frac{d\mathbb{Q}_K}{d\mathbb{P}}\right)+y\left(x+\mathbb{E}^{\mathbb{Q}_K}[h(Y_T)]\right)\right]\right\} \\ &\leq \frac{1}{1-R}\left(\exp\left(\frac{\mu^2}{2\sigma^2 R}\frac{1-R}{R}T\right)\left[1+\frac{1}{2}(1-R)q^2\left(\mathbb{E}^{\hat{\mathbb{P}}}[\langle M\rangle_{T_K}]-\varepsilon\mathbf{1}_{R>1}\right)+O(q^4)\right]\right)^R \\ &\quad \left(x+q\left(\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)]-qRx(\mathbb{E}^{\hat{\mathbb{P}}}[\langle M\rangle_{T_K}]-\varepsilon)+o(q)\right)\right)^{1-R} \\ &\leq u^1_{U_R}(x,q;h)+C\varepsilon q^2, \end{aligned}$$

for some constant C, where we additionally used (3.2.20), (3.2.19) and the derived representations of the measure changes $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}$ and $\frac{d\mathbb{Q}_{\min}}{d\mathbb{P}}$ respectively.

□

Given that we have derived an expansion of the optimal strategy and value function up to order q resp. q^2, we can derive the average utility indifference price $p_{U_R}(x,q;h)$ up to order q:

Corollary 3.2.1. ([Hen02, Theorem 4.2]) *In the setting of Theorem 3.2.1, the average utility indifference price $p_{U_R}(X_t,q;h)$ at time t per unit for q units of $h(Y_T)$ is given by*

$$\begin{aligned} p_{U_R}(X_t,q;h) &= \mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]-q\frac{R}{2}\frac{\eta^2}{X_t}(1-\varrho^2)\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_t^T\frac{Y_u^2(C_u^Y)^2}{(X_u^0/X_t)^2}du\right]+o(q) \\ &= \mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]-q\frac{R}{2}\eta^2(1-\varrho^2)\mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_t^T\frac{Y_u^2(C_u^Y)^2}{X_u^0}du\right]+o(q). \end{aligned}$$

Remark 3.2.2. Here we have also the symmetry in ϱ as discussed in Remark 3.2.1 which is clear in this context.

Proof. To find the average utility indifference price $p_{U_R}(X_t,q;h)$, we have to solve

$$u_{U_R}(X_t,q;0)=u_{U_R}(X_t-qp_{U_R}(X_t,q;h),q;h).$$

This can be written by using Theorem 3.2.1, as

$$X_t^{1-R}=(X_t-qp_{U_R})^{1-R}\left[1+\frac{q\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]}{X_t-qp_{U_R}}-\frac{q^2}{2}R\eta^2(1-\varrho^2)\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_t^T\frac{Y_u^2(\bar{C}_u^Y)^2}{(X_u^0)^2}du\right]\right]^{1-R}.$$

The first order term of $p_{U_R}(X_t,q;h)$ is easily seen to be $\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]$. The second order term is more involved. By making the ansatz

$$p=\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]+kq=C_t+kq$$

in above equation and solving for k, while using the fact that $X_t^0=X_t-qp$, one finds the desired result. The change from $\hat{\mathbb{P}}$ to $\mathbb{Q}_{\min}$ is straightforward by (3.2.18). □

Note. If we consider the **marginal price of a derivative**, that is the limiting price as $q\to 0$, we find

(3.2.21) $$\lim_{q\to 0}p_{U_R}(x,q;h)=\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t].$$

We want to point out that this marginal price is independent of the agent's individual risk aversion and is given by the conditional expectation under the minimal martingale measure of the claim, which is equal to the Black-Scholes price.

3.2.2.2 Incomplete Basis Risk Model with Exponential Utility

We record that the canonical example of an exponential utility function is given by

$$U_\alpha(x) = -\frac{1}{\alpha}e^{-\alpha x}.$$

It turns out, that for this utility function, even explicit price formula can be derived. Hence, we do not have to consider the small claim limit.

Theorem 3.2.2. ([Hen02, Theorem 5.1]) *Assume that q and h satisfy Assumption 1. Then the average utility indifference price $p_U(X_t, q; h)$ at time t for q units of $h(Y_T)$ is (explicitly) given by*

$$p_{U_\alpha}(X_t, q; h) = \frac{-1}{\alpha q(1-\varrho^2)} \log \left(\mathbb{E}^{\mathbb{Q}_{\min}} \left[\exp(-q\alpha(1-\varrho^2)h(Y_T)) \right] \right).$$

Proof. This proof is quite involved and needs a lot of calculations. For details, we refer to [Hen02, Section 5]. However, we will see later in Chapter 4 an alternative approach, where we recover this formula again. □

From this, we easily obtain (can be checked by Taylor expansion) the following expansion for small position sizes.

Theorem 3.2.3. ([Hen02, Section 5]) *In the setting of Theorem 3.2.2, we have*

$$p_{U_\alpha}(X_t, q; h) = \mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t] - \frac{\alpha}{2} q(1-\varrho^2) \left[\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)^2|\mathcal{F}_t] - [\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]]^2 \right] + \mathcal{O}(q^2).$$

If we make use of the (local) relation between the absolute risk aversion α and the relative risk aversion R (derived in Remark 2.0.1), namely that $\alpha = \frac{R}{X_t}$, we get

$$p_{U_\alpha}(X_t, q; h) = \mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t] - \frac{R}{2X_t} q(1-\varrho^2) \left[\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)^2|\mathcal{F}_t] - [\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]]^2 \right] + \mathcal{O}(q^2).$$

Remark 3.2.3. Here we see that the average utility indifference price under exponential utility is independent of the wealth X_t at time t.

Corollary 3.2.2. *Also in this case, we have for the marginal price that*

$$\lim_{q \to 0} p_{U_\alpha}(x, q; h) = \mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t].$$

Proof. This is easily seen as a consequence of the expansion for small position sizes. □

In this case, we even have more: [Mon08, Theorem 5] shows, that

$$\lim_{q \to 0} p_{U_\alpha}(x, q; h) = \lim_{\alpha \to 0} p_{U_\alpha}(x, q; h) = \mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t], \tag{3.2.22}$$

hence the marginal price coincides with the average utility indifference price for zero absolute risk aversion and is given by the arbitrage-free Black-Scholes price.

A risk-neutral investor (this is an investor who is indifferent between various payoffs with the same expected value but not the same risk) is willing to pay the arbitrage-free Black-Scholes price. In other words, the risk-neutral investor does not care if the market is complete or not (i.e. if there is unhedgeable risk) and she is willing to pay the price from the complete model.

3.3 Examples

We finish this chapter by giving concrete examples to illustrate the results.

3.3.1 Call Option

3.3.1.1 Optimal Strategy

We consider $h(Y_T) = (Y_T - K)^+$, the example of a **Call option**. By Assumption 1, we must have that $q > 0$, hence we are in the situation of a **long Call option** position. Our goal is to find an optimal strategy of investing into the risky and traded asset S_t, when an agent holds a small position q in $h(Y_T)$. Under the measure $\mathbb{Q}_{\min}$, we have seen that Y_t has drift $\delta = \nu - \frac{\mu\eta}{\sigma}\varrho$ and more explicitly

$$\frac{dY_t}{Y_t} = \delta dt + \frac{\eta\varrho}{\sigma}\left(\sigma dB_t + \mu dt\right) + \eta\sqrt{1-\varrho^2}dW_t.$$

Moreover, $B_t^{\min} := B_t + \frac{\mu}{\sigma}t$ and $W_t^{\min} := W_t$ are $\mathbb{Q}_{\min}$-Brownian motions. Hence we may define

$$\begin{aligned} Z_t^{\min} &:= \varrho B_t^{\min} + \sqrt{1-\varrho^2}W_t^{\min} \\ &= \varrho\left(B_t + \frac{\mu}{\sigma}t\right) + \sqrt{1-\varrho^2}W_t, \end{aligned}$$

which is clearly a $\mathbb{Q}_{\min}$-Brownian motion correlated to $B_t^{\min}$ and $W_t^{\min}$. With this, the dynamics of Y_t reduce to

$$\frac{dY_t}{Y_t} = \delta dt + \eta Z_t^{\min},$$

and hence, still under $\mathbb{Q}_{\min}$

$$Y_T = Y_0 \exp\left(\left(\delta - \frac{1}{2}\eta^2\right)T + \eta Z_T^{\min}\right).$$

By this we obtain with standard arguments a **Black's formula** for finding an explicit formula for the value of the claim at time $0 \leq t \leq T$

$$\begin{aligned} C_t &= \mathbb{E}^{\mathbb{Q}_{\min}}\left[(Y_T - K)^+ \middle| \mathcal{F}_t\right] \\ &= e^{\delta(T-t)}Y_0\Phi(d_1) - K\Phi(d_2), \end{aligned}$$

for $d_{1,2} = \frac{\log\left(\frac{Y_0}{K}\right) + \delta(T-t) \pm \frac{1}{2}\eta^2(T-t)}{\eta\sqrt{T-t}}$.

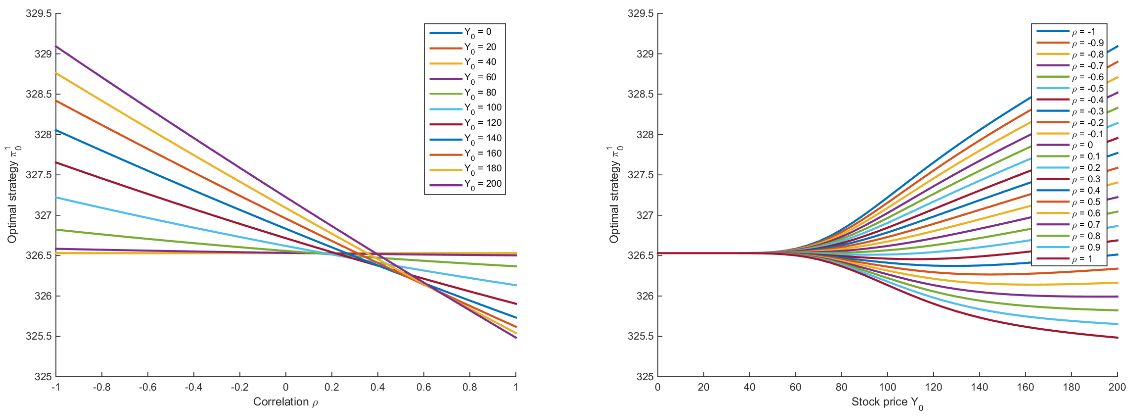
We have seen in Theorem 3.2.1 that the optimal strategy $\tilde{\pi}^\star$, assuming power law utility, is given by

$$\begin{aligned}\tilde{\pi}_t^\star &= \tilde{\pi}_t^1 + o(q)\\ &= \frac{\mu}{\sigma^2 R}(x + qC_t) - \frac{\eta\varrho}{\sigma} qY_t C_t^Y + o(q)\\ &= \frac{\mu}{\sigma^2 R}x + qe^{\delta(T-t)}Y_t\Phi(d_1)\left[\frac{\mu}{\sigma^2 R} - \frac{\eta\varrho}{\sigma}\right] - \frac{\mu}{\sigma^2 R}qK\Phi(d_2) + o(q),\end{aligned}$$

where we used that $C_t^Y = e^{\delta(T-t)}\Phi(d_1)$. We can now illustrate this results using concrete parameters and visualize the results.

Parameters - We choose the following parameters:
$q = 0.01$, $T = 1$, $t = 0$, $K = 100$, $x = 500$, $R = 0.5$, $\mu = 0.04$, $\sigma = 0.35$, $\nu = 0.03$, $\eta = 0.30$.

We see in Figure 3.3.1, that we have a linear dependence between $\tilde{\pi}_0^1$ and ϱ if we fix Y_0. However, given a fixed correlation ϱ, we don't have this linearity anymore, which is clear from above formula. Note that the drift $\delta = \nu - \varrho\frac{\mu\eta}{\sigma}$ of Y_t under $\mathbb{Q}_{\min}$ varies for different ϱ. For $\varrho = 0.875$, we have that $\delta = 0$.



(a) Plot of $\tilde{\pi}_0^1$ for different values of Y_0.

(b) Plot of $\tilde{\pi}_0^1$ for different values of ϱ.

Figure 3.3.1: Plot of the optimal hedge $\tilde{\pi}_0^1$ for a Call option.

In Figure 3.3.2, we take into account the effect of ϱ and Y_0 simultaneously. We observe that if the Call option is far out-of-the-money (i.e. $Y_0 \ll K = 100$), then the optimal strategy $\tilde{\pi}_0^1$ is given by the (constant) Merton hedging strategy $\tilde{\pi}_0 = \frac{\mu}{\sigma^2 R}x$, which is, for our choice of the parameters, $\frac{\mu}{\sigma^2 R}x \approx 326.53$. As the option becomes closer to being in-the-money, $60 \leq Y_t \leq K$ say, the optimal strategy already changes significantly as C_t gets involved. This is of course due to the positive probability that the option can turn in-the-money until time $T = 1$ and hence the agent has to hedge this risk away. Depending on the sign of ϱ, the optimal strategy in- resp. decreases as the option becomes farer in-the-money. However, this is plausible as the agent acts optimally, i.e. she hedges her risk away by taking in a position in S_t. For example, in the case of $\varrho = 1$, there is only one source of risk, hence we are in a situation of a complete market framework and the investor can hedge herself perfectly with a Delta hedge in S_t (which is a short position) and this leads to a decrease of the overall amount of money put into the risky asset S_t.

Lastly, we investigate the sensitivity of π_0^1 with respect to the drift μ. Optimal strategies relative to the Merton hedge for two different values of μ are plotted in Figure 3.3.3. In the case of $\mu = 0$, the

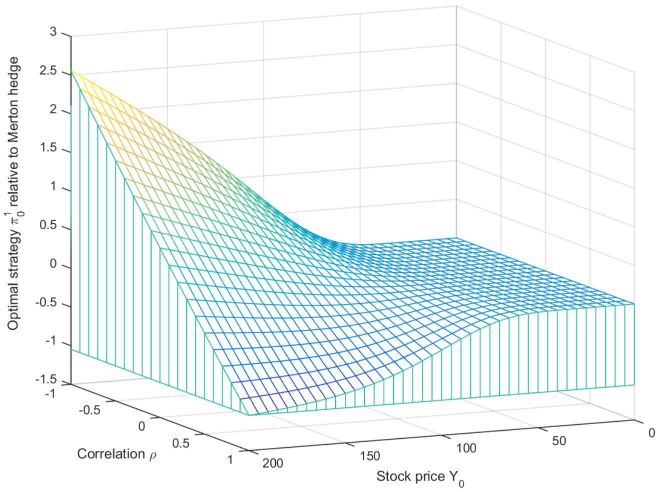
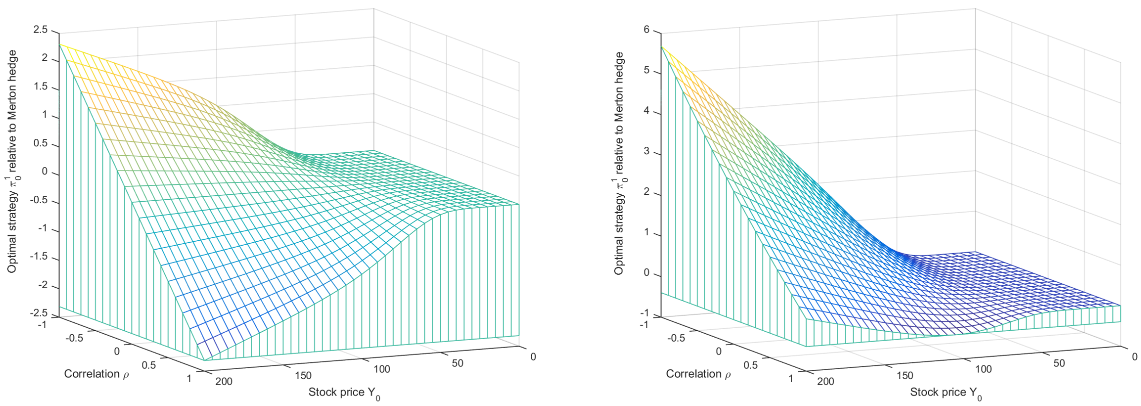


Figure 3.3.2: Plot of $\tilde{\pi}_0^1$ relative to Merton hedge for a Call option.

hedging strategy is only given by $\tilde{\pi}_0^1 = -\frac{\eta\varrho}{\sigma} q e^{\delta(T-t)} Y_t \Phi(d_1)$, hence exactly by the Delta hedging strategy (adjusted for investing into S_t) from the Black-Scholes model. We point out that the larger μ, the more we have the effect that the quantities invested into S_t for hedging Call options being in-the-money are increased. The reason is seen in the fact that the constant proportion $\frac{\mu}{\sigma^2 R}$ of current wealth (which is $X_t + qC_t$) increases. By subtracting the constant Merton hedge $\frac{\mu}{\sigma^2 R} X_t$, we see that the part with the option price gets a higher weight and especially for Call options being in-the-money, the hedging strategy increases.



(a) Optimal strategy $\tilde{\pi}_0^1$ for $\mu = 0$.

(b) Optimal strategy $\tilde{\pi}_0^1$ for $\mu = 0.1$.

Figure 3.3.3: Plot of optimal hedge $\tilde{\pi}_0^1$ for two different drifts for a Call option

3.3.1.2 Pricing

Our next interest lies in the average utility indifference price $p_U(x, q; h)$ at time $t = 0$ which is given by

$$p_U(x, q; h) = \mathbb{E}^{\mathbb{Q}_{\min}}\left[(Y_T - K)^+\right] - q\frac{R}{2}\eta^2(1-\varrho^2)\mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^T \frac{Y_u^2(C_u^Y)^2}{X_u^0}du\right] + o(q).$$

For the first term, we just derived the explicit expression

$$C_0 = e^{\delta T}Y_0\Phi(d_1) - K\Phi(d_2)$$

for d_1, d_2 as given above, whereas for the second term (we omit the factor q)

$$\frac{R}{2}\eta^2(1-\varrho^2)\mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^T \frac{Y_u^2(C_u^Y)^2}{X_u^0}du\right]$$

this is no longer possible and hence we will simulate this based on the known representations of the terms inside the integral.

Let us again record the chosen parameters.

Parameters - We choose the following parameters:
$q = 0.01$, $T = 1$, $t = 0$, $K = 100$, $x = 500$, $R = 0.5$, $\varrho = 0.8$, $\mu = 0.04$, $\sigma = 0.35$, $\nu = \varrho\frac{\eta\mu}{\sigma}$, $\eta = 0.30$.

Note, that for simplification we choose ν such that the drift of Y_t under $\mathbb{Q}_{\min}$ is zero. Indeed, by this choice of parameters, we get $\nu \approx 0.027$, close to the choice in above paragraph.

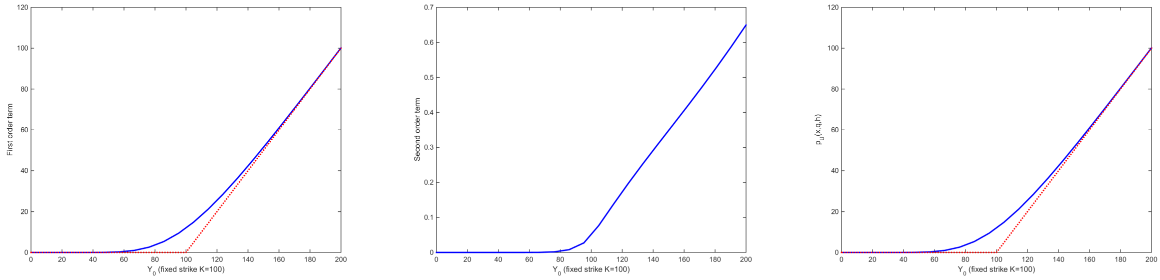
Using above parameters with a fixed correlation of $\varrho = 0.8$, we get for an at-the-money Call option for the first order term the value of 11.9235 while the second order term is approximately given by 0.0477, i.e. 0.4% of the first. Taking the factor q in front of the second order term into account leads us to a percentage of around $q * 0.4\% = 0.004\%$ of the first order term. Hence, the second order term is for very small position sizes clearly negligible and even for $q = 1$, the deviation from the first order term prices is petty.

Note that the first order term is nothing else than the risk-neutral price from the complete model (i.e. the Black-Scholes price) with the a priori fixed parameters as we assume $\delta = 0$, hence purely independent of ϱ and q.

Therefore we see in Figure 3.3.4 the typical Black-Scholes price dynamics of a long position in a Call option given by the first order term, whereas we see in the middle the dynamics of the second order term, which has a negative impact on the first order term. Everything aggregated is seen in the right plot and represents the utility indifference price in the small claim limit. For comparison reasons, we artificially included the payoff profile at time T by the dotted line.

Paying special attention to the second order term, we point out that in the case of increasing Y_0, the second order term has nearly a linear growth characteristic on the interval $[0, 200]$ with a very small slope [14]. The price dynamics with respect to the underlying process in the area of $Y_0 \leq 90$ are very close to the classical Black-Scholes price dynamics and the price spread increases as the option gets farer in-the-money.

[14] However, it turns out that the second order term has a quadratic growth behavior.



(a) First order term. (b) Second order term. (c) $p_U(x,q;h)$.

Figure 3.3.4: Different price plots with varying Y_0 for a Call option.

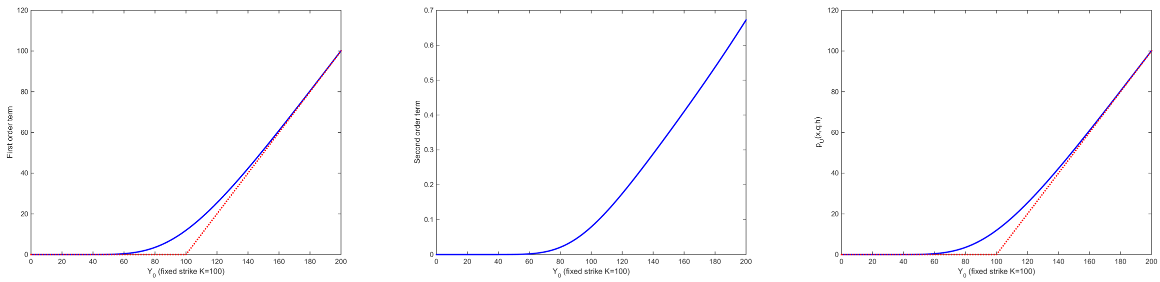
Moreover, we also provide the price dynamics for exponential utilities, where we use the (local) relation of $\alpha = \frac{R}{x_0}$ as already seen.
We get for the value of the Call option at time $t = 0$

$$p_{U_\alpha}(x,q;h) = C_0 - q\frac{R}{2x}(1-\varrho^2)\left[\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)^2] - C_0^2\right],$$

where

$$\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)^2] = Y_0^2 \exp\left(2\delta T + \eta^2 T\right)\Phi(d_1 + \eta\sqrt{T}) - 2KY_0\Phi(d_1) + K^2\Phi(d_2).$$

Everything aggregated is plotted in Figure 3.3.5 (we omit the factor q for plotting the second order term).



(a) First order term. (b) Second order term. (c) $p_U(x,q;h)$.

Figure 3.3.5: Different price plots with varying Y_0 for a Call option under exponential utility.

It can be concluded that, independently of the utility function, the results are roughly on par. This is clear, as the first order term is purely independent of the individual utility function and the second order terms are relatively small.

However, the attentive reader notices that the second order term for exponential utility is sparsely larger. This is of course not obvious by what we have seen so far and we try to explain this heuristically.
We have the following relationship for the two considered utility functions, when risk aversion goes to

zero [15]

$$\lim_{\alpha \to 0} \frac{1 - e^{-\alpha x}}{\alpha} = x = \lim_{R \to 0} \frac{x^{1-R}}{1-R}.$$

As power utility is only defined on the positive real line while exponential utility is defined on the whole real line, there is no reason why we should expect the same limiting behavior for the risk aversion parameter converging to zero. Therefore, we consider Figure 3.3.6, which consists of the second order terms with respect to R (and $\frac{\alpha}{x}$).
We see here that for a value of $R = 0.5$, the difference is approximately given by 0.02, of course also negligible.

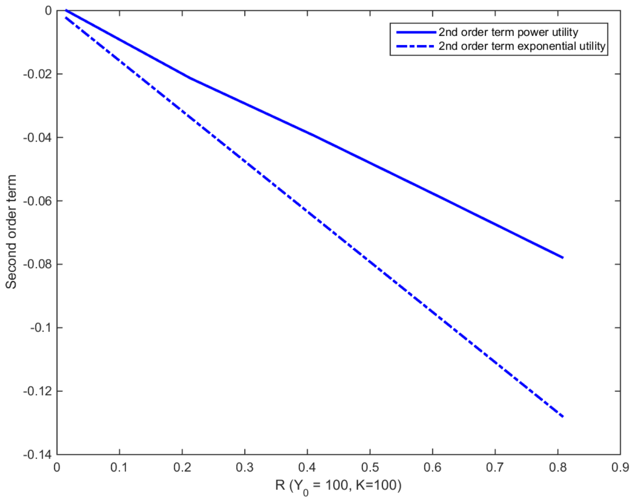


Figure 3.3.6: Plot of the second order terms with varying risk aversion R for an ATM Call option.

Lastly, Figure 3.3.7 shows simultaneously the dynamics of the average utility indifference price $p_U(x, q; h)$ with respect to both the correlation ϱ as well as the position size q near zero.
Of course, for $\varrho = 1$, we recognize the arbitrage-free price $p \approx 11.9235$ from the complete model. We can also see that the area around $(\varrho = 1, q = 0)$ is quite robust and does not differ much from the arbitrage-free price. But as the position size grows, the resulting negative second order term gets a higher weight and hence a resulting higher impact.
Moreover, independently of the growth rate of q and ϱ, the resulting limiting price is always the arbitrage-free Black-Scholes price. This is not anymore the case in the large claim limit as treated in Chapter 4.

[15] Note that the proper definition of exponential utility including zero absolute risk aversion is given by $U_\alpha(x) = \frac{1-e^{-\alpha x}}{\alpha}$ and $U_0(x) = x$. As we never have dealt with zero absolute risk aversion before, our initial definition of $U_\alpha(x) = -\frac{1}{\alpha}e^{-\alpha x}$ was not false as they only differ, for positive fixed absolute risk aversion, by a constant. We have seen that utility functions are unique up to linear transformations.

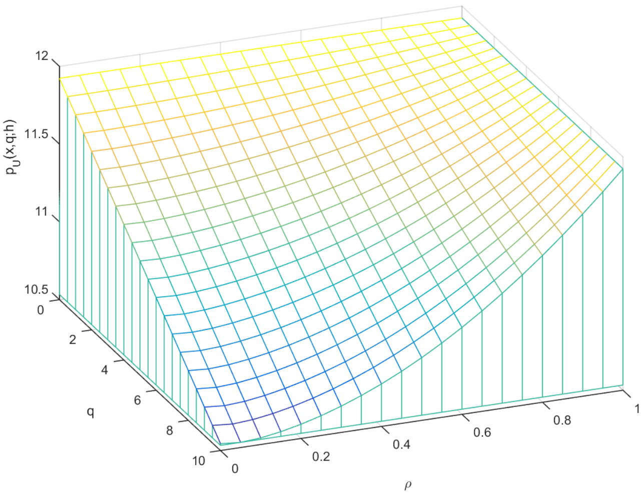


Figure 3.3.7: $p_U(x, q; h)$ with respect to ϱ and q for a Call option.

3.3.2 Put Option

The very same procedure can be applied for a Put option.

3.3.2.1 Optimal Strategy

Consider $h(Y_T) = (K - Y_T)^+$. We may assume that $q > 0$, hence we are in the setting of a **long Put option** position.

Parameters - We choose the following parameters:
$q = 0.01$, $T = 1$, $t = 0$, $K = 100$, $x = 500$, $R = 0.5$, $\mu = 0.04$, $\sigma = 0.35$, $\nu = 0.03$, $\eta = 0.30$.

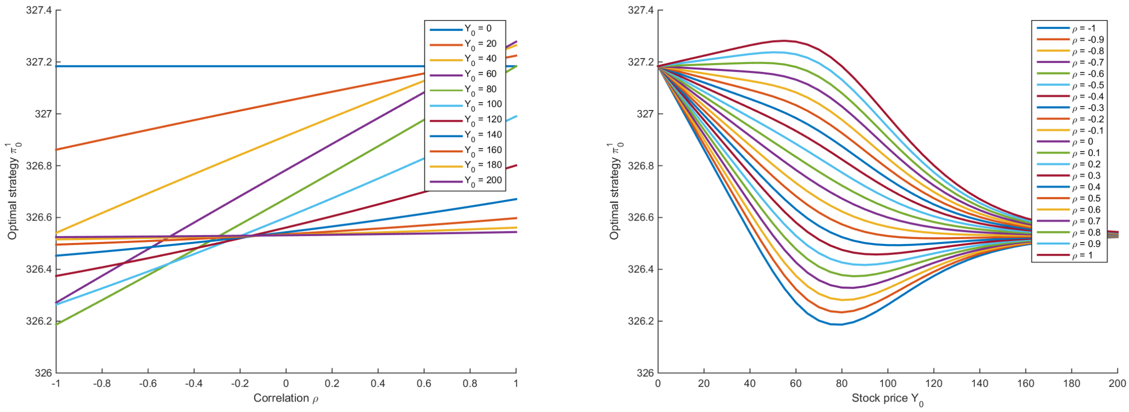
We get

$$\begin{aligned} C_t &= \mathbb{E}^{\mathbb{Q}_{\min}}\left[(K - Y_T)^+\right] \\ &= K\Phi(-d_2) - Y_0 e^{\delta(T-t)}\Phi(-d_1), \end{aligned}$$

for $d_{1,2} = \frac{\ln\left(\frac{Y_0}{K}\right)+\delta(T-t)\pm\frac{1}{2}\eta^2(T-t)}{\eta\sqrt{T-t}}$.
This gives

$$\tilde{\pi}_t^1 = \frac{\mu}{\sigma^2 R}x + qY_t e^{\delta(T-t)}\Phi(-d_1)\left[-\frac{\mu}{\sigma^2 R} + \frac{\eta\varrho}{\sigma}\right] + \frac{\mu}{\sigma^2 R}qK\Phi(-d_2).$$

Figure 3.3.8 shows that as the option is in-the-money, the optimal strategy increases when compared with the constant Merton hedge $\frac{\mu}{\sigma^2 R}x \approx 326.53$. This is of course due to the fact that the Delta hedge is a long position in S_t.
Everything aggregated and relative to the Merton hedge, we see in Figure 3.3.9 that as long as the correlation is positive, the optimal strategy consists of the Merton hedge plus an additional positive position in S_t, while in the other case, the hedge becomes a short position, hence has an overall negative impact on the optimal strategy.



(a) Plot of π_t^1 for different values of Y_t.

(b) Plot of π_t^1 for different values of ϱ.

Figure 3.3.8: Plot of the optimal hedge $\tilde{\pi}_t^1$ for a Put option.

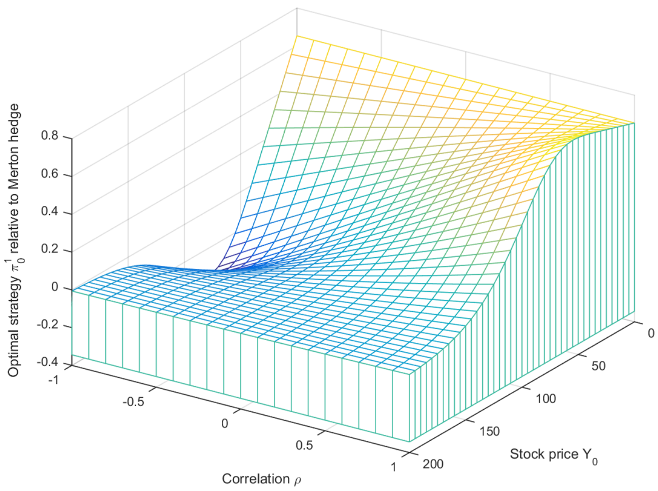


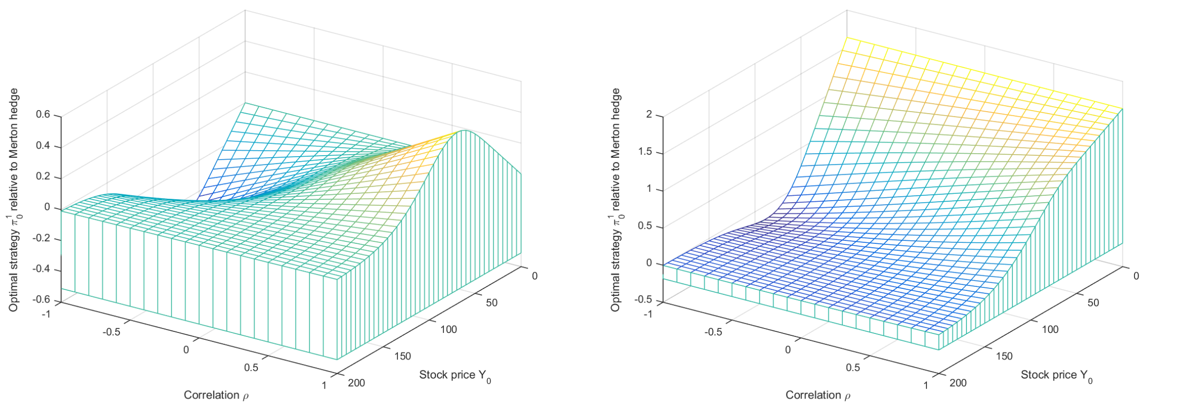
Figure 3.3.9: Plot of $\tilde{\pi}_0^1$ relative to Merton hedge for a Put option.

Moreover, we also investigate Figure 3.3.10 the behavior of the optimal hedging strategy $\tilde{\pi}_0^1$ for different μ. Also here, for $\mu = 0$, we recover the classical Delta hedging strategy (modified for investing into S_t) from Black-Scholes model. Additionally, the strategies for Put options being in-the-money increase by increasing the drift μ. Seeing the reason, we can adopt the argument as given in the example of the Call option.
We turn our attention to the pricing dynamics.

3.3.2.2 Pricing

Again, our next interest lies in the average utility indifference price $p_U(x, q; h)$ which is given by

$$p_U(x, q; h) = \mathbb{E}^{\mathbb{Q}_{\min}}\left[(K - Y_T)^+\right] - q\frac{R}{2}\eta^2(1-\varrho^2)\mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^T \frac{Y_u^2 (C_u^Y)^2}{X_u^0} du\right] + o(q).$$



(a) Optimal strategy $\tilde{\pi}^1_0$ for $\mu = 0$. (b) Optimal strategy $\tilde{\pi}^1_0$ for $\mu = 0.1$.

Figure 3.3.10: Plot of optimal hedge $\tilde{\pi}^1_0$ for two different drifts for a Put option

The first term is given by

$$C_t = K\Phi(-d_2) - e^{\delta(T-t)}Y_0\Phi(-d_1),$$

for d_1, d_2 as given above, hence

$$C^Y_u = -e^{\delta(T-u)}\Phi(-d_1).$$

By this, the second order term is given by (we omit the factor q)

$$\frac{R}{2}\eta^2(1-\varrho^2)\mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^T \frac{Y_u^2(C^Y_u)^2}{X^0_u}du\right].$$

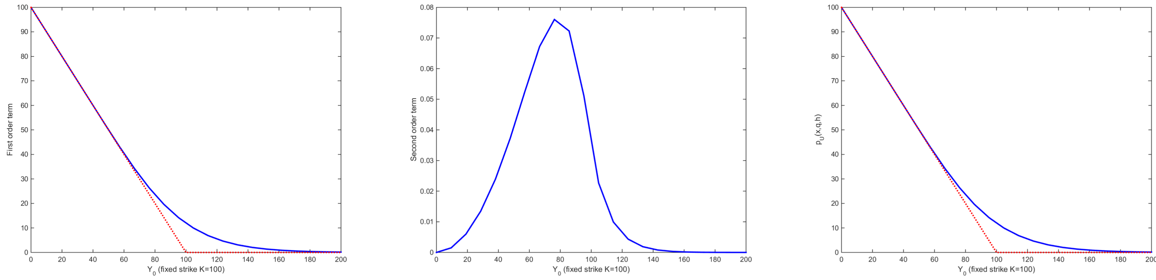
Parameters - We choose the following parameters:
$q = 0.01$, $T = 1$, $t = 0$, $K = 100$, $x = 500$, $R = 0.5$, $\varrho = 0.8$, $\mu = 0.04$, $\sigma = 0.35$, $\nu = \varrho\frac{\eta\mu}{\sigma}$, $\eta = 0.30$.

In Figure 3.3.11 we simulate the first resp. second order term and the aggregated average utility indifference price $p_U(x, q; h)$. Also here, we artificially included the payoff structure at time T for a long position in the Put option.
For one unit of a Put option with $Y_0 = 80$ and $K = 100$, we get a value of 23.5344 for the first order term, whereas the second order term is now 0.0764, hence approximately 0.32% of the first order term. Also here, the second order term is clearly negligible for small position sizes.
Also for the Put option, we provide the price dynamics under exponential utility, see Figure 3.3.12. In this case, the average utility indifference price is given by

$$p_{U_\alpha}(x_0, q; h) = C_0 - q\frac{R}{2x_0}(1-\varrho^2)\left[\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)^2] - C_0^2\right],$$

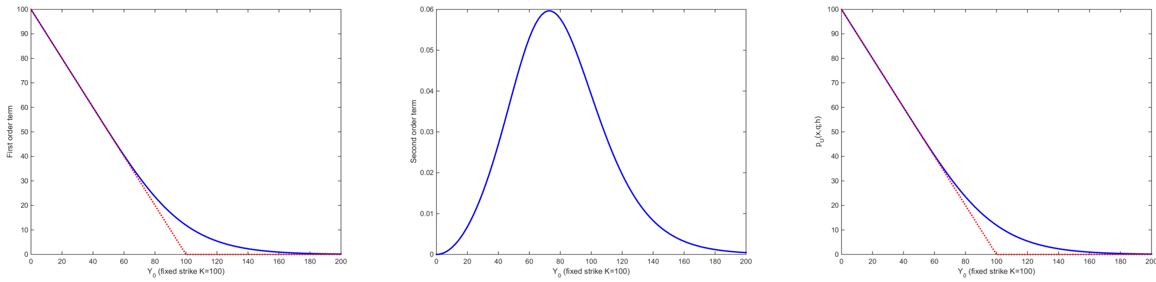
where

$$\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)^2] = K^2\Phi(-d_2) - 2KY_0\Phi(-d_1) + Y_0^2\exp((2\delta+\eta^2)T)\Phi(-d_1 - \eta\sqrt{T}).$$



(a) First order term. (b) Second order term. (c) $p_U(x, q; h)$.

Figure 3.3.11: Different price plots with varying Y_0 for a Put option.



(a) First order term. (b) Second order term. (c) $p_U(x, q; h)$.

Figure 3.3.12: Different price plots with varying Y_0 for a Put option under exponential utility.

The rough dynamics look similar and the second order term under exponential utility is in this case a slightly smaller than the one under power utility. The reasoning can be adapted from the previous example and is visualized in Figure 3.3.13.

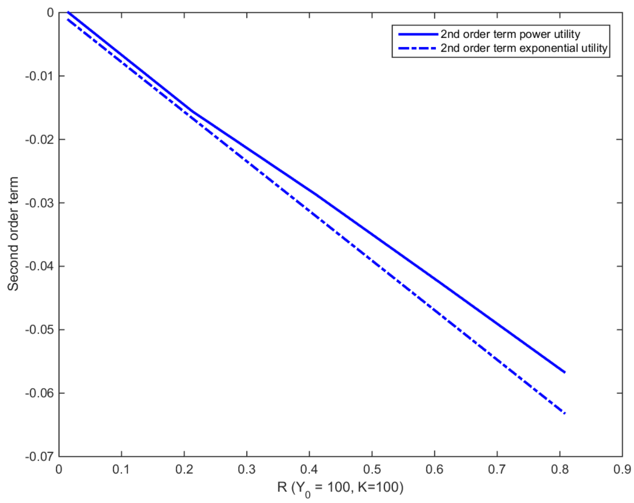


Figure 3.3.13: Plot of the second order terms with varying risk aversion R for an ATM Put option.

The difference for $R = 0.5$ and $Y_0 = K = 100$ is approximately 0.005, which coincides with the difference seen when comparing Figure 3.3.11 with Figure 3.3.12. But as the second order term is almost negligible, this small difference does not have any impact on the utility indifference price.

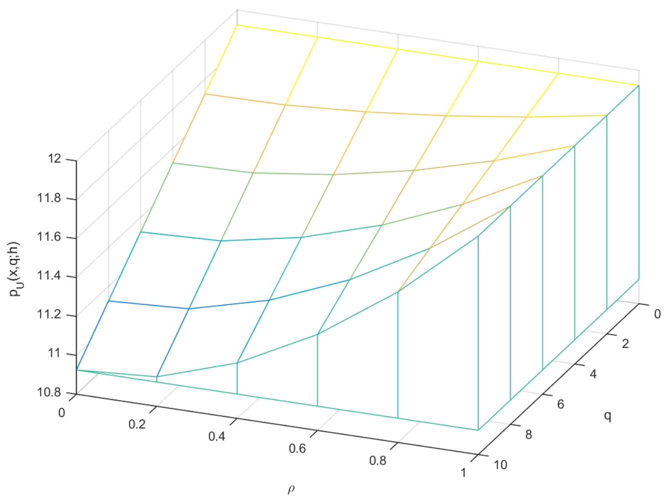


Figure 3.3.14: $p_U(x, q; h)$ with respect to ϱ and q for a Put option

In Figure 3.3.14, we see the price dynamics with respect to ϱ and q simultaneously. Also in that case, it turns out that, independently of the rate of convergence of q and ϱ, the limiting price is always given by the arbitrage-free Black-Scholes price. This phenomenon is no longer true in the large claim limit.

3.3.3 Power Option

3.3.3.1 Optimal Strategy

Consider the non-standard claim $h(Y_T) = Y_T^2$. This is a financial instrument, that pays at maturity T the holder the square of the stock value at time T. Note that in either case, there will be a payout, which was not the case of before presented example of a Call and Put option respectively. Again by Assumption 1, we require that $q > 0$. We then get

$$\begin{aligned} C_0 &= \mathbb{E}^{\mathbb{Q}_{\min}}\left[Y_T^2\right] \\ &= \mathbb{E}^{\mathbb{Q}_{\min}}\left[\left(Y_0 \exp\left(\left(\delta - \frac{1}{2}\eta^2\right)T + \eta Z_T^{\min}\right)\right)^2\right], \end{aligned}$$

which reduces to

$$C_0 = Y_0^2 \exp\left(2\delta T + \eta^2 T\right),$$

and more generally to

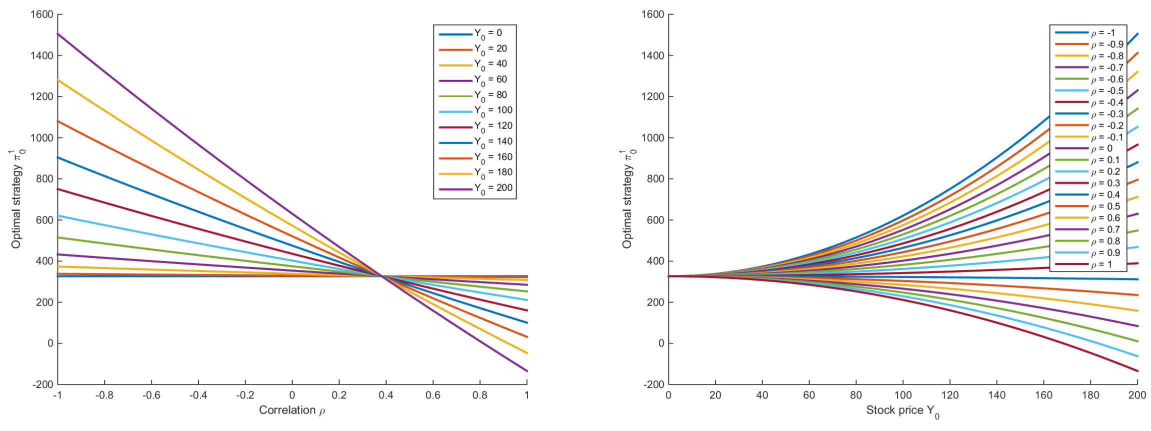
$$C_t = Y_0^2 \exp\left(2\delta(T-t) + \eta^2(T-t)\right).$$

By this, we obtain the optimal strategy as

$$\tilde{\pi}_t^\star = \frac{\mu}{\sigma^2 R}x + qY_t^2 \exp\left((2\delta + \eta^2)(T-t)\right)\left[\frac{\mu}{\sigma^2 R} - 2\frac{\eta\varrho}{\sigma}\right] + o(q).$$

Let us specify the parameters used.

Parameters - We choose the following parameters:
$q = 0.01$, $T = 1$, $t = 0$, $K = 100$, $x = 500$, $R = 0.5$, $\mu = 0.04$, $\sigma = 0.35$, $\nu = 0.03$, $\eta = 0.30$.



(a) Plot of $\tilde{\pi}_t^1$ for different values of Y_t.

(b) Plot of $\tilde{\pi}_t^1$ for different values of ϱ.

Figure 3.3.15: Plot of the optimal hedge $\tilde{\pi}_t^1$ for a non-standard claim.

In Figure 3.3.15 resp. Figure 3.3.16 we see the hedging behavior of the agent. In the extreme scenario of

$\varrho = -1$ and $Y_0 = 200$, the investor should take in a position in S_t that exceeds by far our initial wealth of $x = 500$ (~ 1400).

This is reasonable as if an investor holds $q = 0.01$ units of $h(Y_T) = Y_T^2$, and Y_0 is already at level of 200, then she takes in a huge position in S_t due to the fact that she wins in either the position $qh(Y_T)$ or in S_t. In contrast to that, she acts in the opposite direction if $\varrho = 1$ and takes in even a short position.

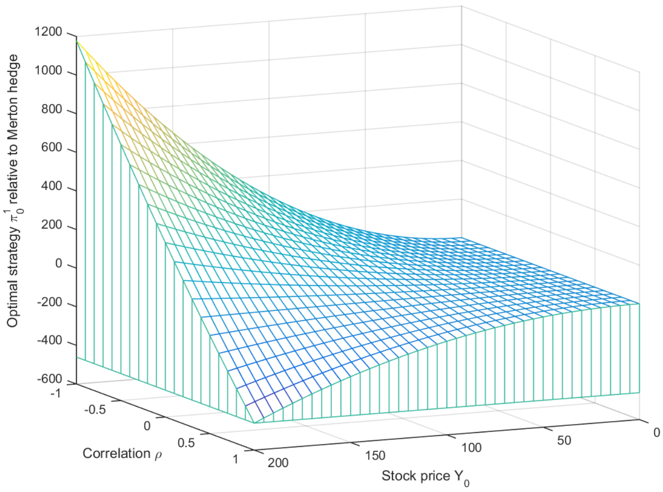
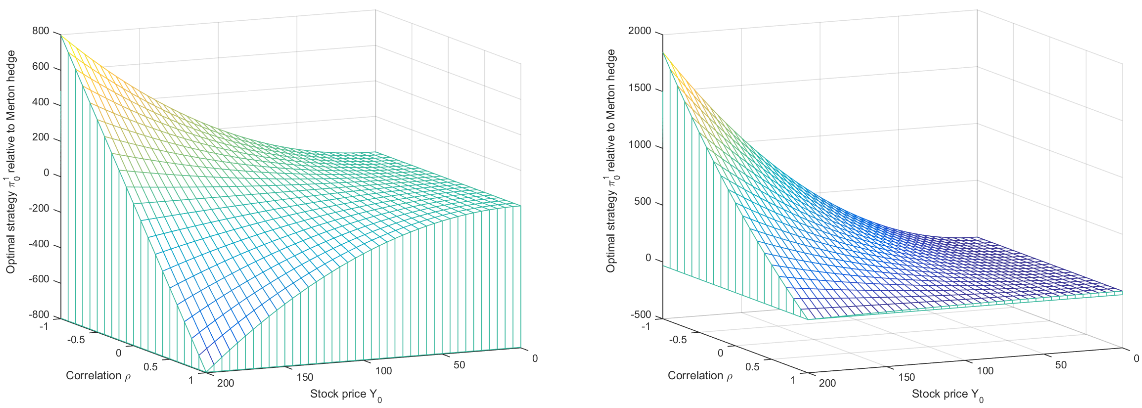


Figure 3.3.16: Plot of the optimal hedge $\tilde{\pi}_t^1$ relative to Merton hedge for a non-standard claim.



(a) Optimal strategy $\tilde{\pi}_0^1$ for $\mu = 0$.

(b) Optimal strategy $\tilde{\pi}_0^1$ for $\mu = 0.1$.

Figure 3.3.17: Plot of optimal hedge $\tilde{\pi}_0^1$ for two different drifts for a non-standard claim.

Moreover, in Figure 3.3.17, we present hedging strategies with respect to two different values of the drift μ. Figure 3.3.17 (a) shows the typical hedging strategy from the Black-Scholes model, as we chose $\mu = 0$. On the other hand, we present in Figure 3.3.17 (b) the hedging strategies for the drift $\mu = 0.1$. Also here, we note that the larger μ, the higher the amount of money put into S_t due to the reason that the constant fraction $\frac{\mu}{\sigma^2 R}$ of money invested into S_t increases.

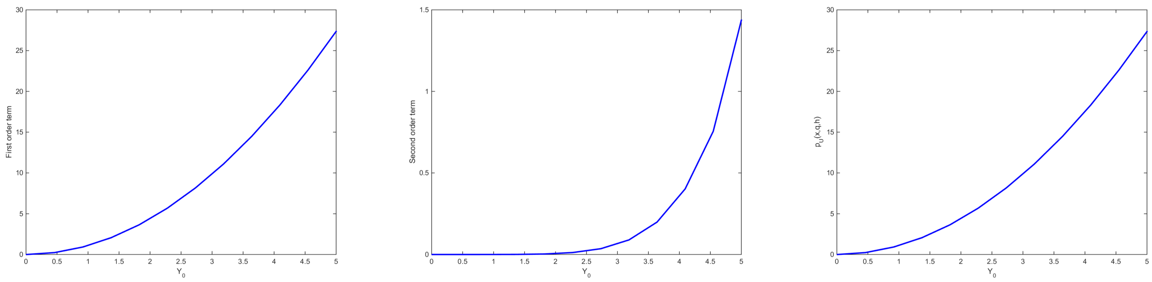
3.3.3.2 Pricing

We are now interested in pricing such an instrument. The average utility indifference price is given by

$$p_U(x,q;h) = \mathbb{E}^{\mathbb{Q}_{\min}}\left[Y_T^2\right] - q\frac{R}{2}\eta^2(1-\varrho^2)\mathbb{E}^{\mathbb{Q}_{\min}}\left[\int_0^T \frac{Y_u^2(C_u^Y)^2}{X_u^0}du\right] + o(q),$$

where $C_t^Y = 2Y_0\exp((2\delta+\eta^2)(T-t))$. We have seen an explicit formula for the first term, whereas for the second term, we again simulate it. Due to the power-like behavior of the optimal strategy, we focus on the interval $Y_0 \in [0,5]$.

Parameters - We choose the following parameters:
$q = 0.01$, $T = 1$, $t = 0$, $K = 100$, $x = 500$, $R = 0.5$, $\varrho = 0.8$, $\mu = 0.04$, $\sigma = 0.35$, $\nu = \varrho\frac{\eta\mu}{\sigma}$, $\eta = 0.30$.



(a) First order term. (b) Second order term. (c) $p_U(x,q;h)$.

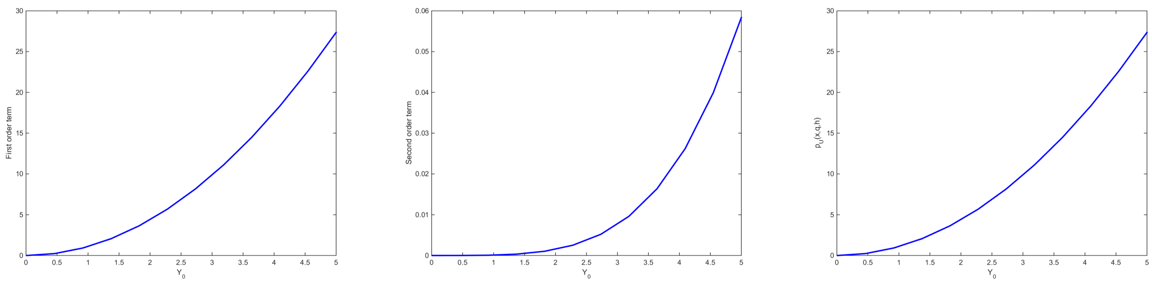
Figure 3.3.18: Different price plots with varying Y_0 for a non-standard payoff.

Again, we provide the price dynamics under exponential utility, see Figure 3.3.19. We have that

$$p_{U_\alpha}(x_0,q;h) = C_0 - q\frac{R}{2x_0}(1-\varrho^2)\left[\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)^2] - C_0^2\right],$$

where

$$\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)^2] = \mathbb{E}^{\mathbb{Q}_{\min}}[Y_T^4] = Y_0^4\exp((4\delta+6\eta^2)T).$$



(a) First order term. (b) Second order term. (c) $p_U(x,q;h)$.

Figure 3.3.19: Different price plots with varying Y_0 for a non-standard payoff under exponential utility.

Figure 3.3.18 and Figure 3.3.19 show the pricing dynamics of the Power option. In this case, the difference between the two second order terms is enormous. Figure 3.3.20 shows the spread between the two second order terms for Y_0 with respect to the risk aversion.

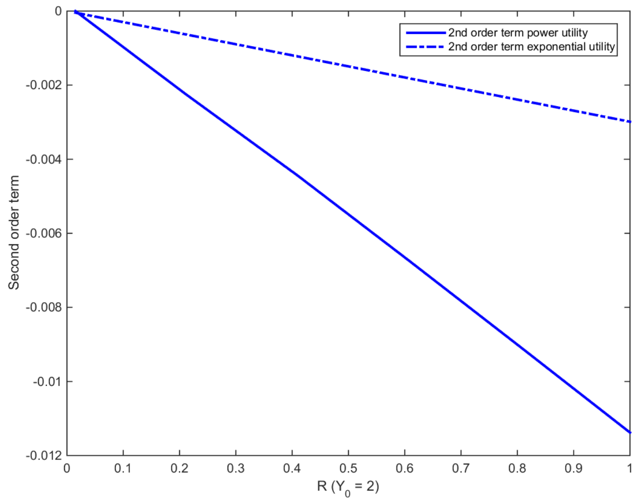


Figure 3.3.20: Plot of the second order terms with varying risk aversion R for Power option ($Y_0 = 2$).

3.4 Conclusion

We have studied in the small claim limit the somehow *classical* approach to utility indifference pricing of claims on a nontraded asset Y_t. *Classical* is meant in the sense of *intuitive* as we considered the intuitive idea of taking into account closely related assets to protect oneself from the arising risk.

In practice, this approach is widely used and accepted. For instance, pricing a financial instrument on some family (often called basket) of shares (e.g. in form of a structured product), one can treat each single share as nontradable due to high transaction costs and one can use a closely related index for pricing and hedging this claim.

Rigorously speaking, we started with the Classical Black-Scholes-Merton Model and then artificially added a closely related, nontraded asset to the model. In the complete case, where this close relation is indeed perfect, we presented that this adjusted model is nothing else than a classical Black-Scholes-Merton problem with a modified strategy. This modification arises as the agent hedges herself perfectly from the risk arising from $h(Y_T)$ by a Delta hedge in S_t. In this scenario, the claim $h(Y_T)$ has a unique, utility-independent price which is given by the arbitrage-free price from classical valuation in complete markets.

If the market is not anymore complete, in order to specify a particular price for a claim and optimal hedging strategies, we have to introduce the agent's individual aversion towards risk by specifying her individual utility function. We considered such with either constant relative risk aversion (i.e. power utility) or constant absolute risk aversion (i.e. exponential utility).

In the case where there is some non-vanishing hedging error, we provided a first order approximation for the optimal strategy in a neighborhood of $q = 0$, which is, surprisingly, given by a slightly adjusted version of the optimal strategy derived from the complete case in the sense that the second order term (for Delta hedging) gets less involved as the magnitude of ϱ decreases. By specifying the optimal strategy,

we implicitly constructed an optimal hedging strategy. We then have seen that the modification of the optimal strategy has an interesting interpretation - it is due to the fact that the agent protects herself with a (not anymore perfect) Delta hedge.
By this we also derived an expression for the value function up to order q^2 and even explicit formulae for average utility indifference prices up to the same order.
Also under exponential utility, we presented an explicit pricing formula. This is an advantage of the considered utility function: Calculations become more tractable and in some cases even explicit formulae (without forced to consider the small claim limit) can be derived. However, we also provided an expansion of the value function around $q = 0$.
Lastly, we have applied our results to concrete examples in the case of power law utility and exponential utilities. We have studied optimal strategies and average utility indifference prices in the case of Call, Put and Power options. Surprisingly, the main driver for determining prices lies in the first order term which is the arbitrage-free Black-Scholes price. In the example of a Call option ($q = 0.01$), the second order term is 0.004% of the first order term, hence clearly negligible.
A disadvantage of this approach lies in the disability of pricing short Call options. This is due to the fact that we needed some assumptions to ensure that we do not get into trouble with our mathematical development.

All these results were studied in the limit as $q \to 0$, i.e. in the small claim limit. We shall see in the next chapter that also a large claim limit approach can be applied.

Chapter 4

Large Claim Limit Approach

In this chapter, we present an alternative approach to the one studied in the previous chapter. We will study the value function in the large claim limit, i.e. in the range of large position sizes and high correlations. Our main reference is [Rob13]: Pricing for Large Positions in Contingent Claims (2013), *Mathematical Finance*, Forthcoming.

The approach lies in the study of the behavior of the value function $u_U^n(x, q_n; h^n)$ in the stochastic factor resp. basis risk model as $n \to \infty$ by allowing the markets (i.e. the correlation of the assets) as well as the position size q_n to vary under the constraints of $\varrho_n \to 1$ and $q_n \to \infty$. Economically speaking, we consider prices in a sequence of markets that become asymptotically complete in the limit while the position size grows to infinity. We point out, that the large position sizes can also arise endogenously, in the sense that by the market convergence, roughly speaking, prices will also converge in some sense to a limiting price. If this limiting price is not equal to the arbitrage-free Black-Scholes price, then large position sizes (possibly infinite position sizes) come into play as agents try to make use of this asymptotic arbitrage opportunity.
Moreover, it is important to not restricting the markets to a fixed market as in such a case, by increasing the position size, the unhedgeable component (per unit) poses an overall large risk and therefore only two cases can occur. Either, the agent does not hold a large position as it is too risky or she is only willing to pay the lowest possible (arbitrage-free) price. By allowing the markets to vary, and under the assumption that hedging errors converge to zero (i.e. $\varrho_n \to 1$), we avoid such a scenario. Nevertheless, we will provide results that explain exactly above heuristic arguments mathematically.

In one of our main results, we will investigate the different drivers for the average utility indifference price $p_U^n(x, q_n; h^n)$ in the limit, i.e. as the markets become asymptotically complete due to vanishing hedging errors. Depending on the speed of q_n growing to infinity, we will see that it is worth distinguishing three different market regimes. It turns out that they differ significantly in their limiting price and we will see the so-called large position effect - an effect that enables an investor with a large position size to push prices towards the superreplication price. We then finally examine these results using concrete examples.

Moreover, we investigate the behavior of the difference of two utility indifference prices with the same growth rate for large negative wealths in the large claim limit. It turns out that this difference will vanish in the limit.

Lastly, we will see that in the basis risk model with exponential utility function, the optimal position size in $h(Y_T)$ (in the sense of maximized expected utility at time T) to be taken by the investor satisfies the heuristic relationship given the agent can buy the claims for an arbitrage-free price $p \in I(h)$

$$\text{risk aversion} \times \text{position size} \times \text{hedging error} \approx \text{const.} \tag{4.0.1}$$

Thus, for a given risk aversion, larger positions come along with lower hedging errors and vice versa. Or, put differently, large position sizes can also affect the agent's risk aversion.

It turns out that if position size $\times$ hedging error ≈ 0, then prices converge to the Black-Scholes price. As a consequence, investor would not hold a large position in the claim, therefore this regime can be compared with the small claim limit as established in Chapter 3. Moreover, if position size $\times$ hedging error $\approx$ constant, then the limiting price is given by the canonical exponential utility price. Hence in this case it seems that incompleteness procreates the most and can still observed in the limit. Due to the asymptotic arbitrage opportunity, the large claim limit arises endogenously in this regime. Lastly, if position size $\times$ hedging error explodes, we are in the framework as described previously that the agent is only willing to pay the lowest arbitrage-free price, which is somehow unsatisfactory. But it turns out, that the latter regime won't appear when agents are acting optimally.

Let us first state some auxiliary results which will be needed later in our study.

4.1 Auxiliary Results

In this section, we consider a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions. The first lemma gives us some helpful equivalences on the generalized relative entropy for $U \in \mathcal{U}_\alpha$, resp. $U \in \mathcal{U}_{p,l}$. These will be often used when applying the dual approach.

Lemma 4.1.1. ([Rob13, Lemma A.2]) *Let $Y \geq 0$. Then the following statements are equivalent:*

1) $\mathbb{E}^{\mathbb{P}}[V(yY)] < \infty$ *for all* $\alpha > 0, U \in \mathcal{U}_\alpha$ *and* $y > 0$.

2) $\mathbb{E}^{\mathbb{P}}[V(yY)] < \infty$ *for some* $\alpha > 0, U \in \mathcal{U}_\alpha$ *and* $y > 0$.

3) $\mathbb{E}^{\mathbb{P}}[Y \log(Y)] < \infty$.

Moreover, let $p > 1$ and set $\gamma := \frac{p}{p-1}$. Then the following statements are equivalent:

A) $\mathbb{E}^{\mathbb{P}}[V(yY)] < \infty$ *for all* $l > 0, U \in \mathcal{U}_{p,l}$ *and* $y > 0$.

B) $\mathbb{E}^{\mathbb{P}}[V(yY)] < \infty$ *for some* $l > 0, U \in \mathcal{U}_{p,l}$ *and* $y > 0$.

C) $\mathbb{E}^{\mathbb{P}}[Y^\gamma] < \infty$.

Proof. The main idea of the proof lies in using (2.0.1) resp. (2.0.2) to get explicit formulae for V in terms of V_α resp. V_p.
Let $\alpha > 0$ and $U \in \mathcal{U}_\alpha$. We know by (2.0.1) that $V_\alpha(y) = \frac{y}{\alpha}(\log(y) - 1)$ and that $\lim_{y\to\infty} \frac{V(y)}{V_\alpha(y)} = 1$. Therefore, for every $\varepsilon > 0$, we can find $M = M(\varepsilon, U)$ such that for $y \geq M$, we have the following bounds on $V(y)$ for y lying outside the disk with radius M:

$$\frac{1-\varepsilon}{\alpha} y(\log(y) - 1) \leq V(y) \leq \frac{1+\varepsilon}{\alpha} y(\log(y) - 1).$$

As $|V(y)|$ as well as $|y(\log(y)) - 1)|$ are bounded on the compact set $[0, M]$, there exists a constant $C = C(\varepsilon, M) > 0$ such that we have the following bounds on the whole real line

$$-C + \frac{1-\varepsilon}{\alpha} y \log(y) \leq V(y) \leq C + \frac{1+\varepsilon}{\alpha} y \log(y).$$

We then get the implications 1) $\implies$ 2) trivially, 2) $\implies$ 3) by considering the first inequality and finally 3) $\implies$ 1) by the sandwich principle. This proves the first assertion.
We know by (2.0.2) that for every $\varepsilon > 0$ there exists some constant $M = M(\varepsilon, U)$ such that for $y \geq M$, we have

$$(1-\varepsilon)\hat{l}y^{\gamma} \leq V(y) \leq (1+\varepsilon)\hat{l}y^{\gamma}.$$

Again, by the boundedness of $|V(y)|$ and y^{γ}, we get the existence of a $C > 0$ such that on the whole real line

$$-C + (1-\varepsilon)\hat{l}y^{\gamma} \leq V(y) \leq C + (1+\varepsilon)\hat{l}y^{\gamma}.$$

The equivalences $A) \iff B) \iff C)$ follow in a similar manner as above. ☐

The next lemma shows that under some conditions, we have the differentiability of $y \mapsto \mathbb{E}^{\mathbb{P}}[V(yY)]$ with even surjective derivative. This will later be used among others in a proof of a statement that rules out asymptotic arbitrage.

Lemma 4.1.2. ([Rob13, Lemma A.3]) *Let $\alpha > 0, p > 1, l > 0$. Let $U \in \mathcal{U}_{\alpha} \cup \mathcal{U}_{p,l}$. Furthermore, let $Y \geq 0$ be a random variable with $\mathbb{E}^{\mathbb{P}}[Y] = 1$ and such that $\mathbb{E}^{\mathbb{P}}[V(Y)] < \infty$. Then:*

- *The map $y \mapsto \mathbb{E}^{\mathbb{P}}[V(yY)]$ is differentiable with derivative $\mathbb{E}^{\mathbb{P}}[YV'(yY)]$.*
- *For any $x \in \mathbb{R}$, there exists a unique y such that $\mathbb{E}^{\mathbb{P}}[YV'(yY)] = x$, hence the derivative is surjective.*

Proof. For $\varepsilon > 0$ and $z \geq 0$, we consider

$$f(\varepsilon, z) := \frac{V((y+\varepsilon)z) - V(yz)}{\varepsilon} - \frac{V(yz)}{y}.$$

We first note that $f(\varepsilon, 0) = 0$. As V is convex, V' is strictly increasing, hence

$$\partial_z f(\varepsilon, z) = \frac{y+\varepsilon}{\varepsilon}\left(V'((y+\varepsilon)z) - yV'(yz)\right) \geq 0.$$

Moreover, the convexity of V implies that

$$f(\varepsilon, Y) \leq \frac{\varepsilon V((1+y)Y) + (1-\varepsilon)V(yY) - V(yY)}{\varepsilon} - \frac{V(yY)}{y} = V((1+y)Y) - V(yY) - \frac{V(yY)}{y}.$$

Hence taking the $\mathbb{P}$-expectation of above inequality yields that $\mathbb{E}^{\mathbb{P}}[f(\varepsilon, Y)] < \infty$ for all $\varepsilon > 0, y > 0$ and even for the limit as $\varepsilon \to 0$ due to the previous lemma.
By this, we can apply dominated convergence to $f(\varepsilon, Y)$ yielding that

$$\partial_y \mathbb{E}^{\mathbb{P}}[V(yY)] = \lim_{\varepsilon \to 0} \frac{\mathbb{E}^{\mathbb{P}}[V((y+\varepsilon)Y)] - \mathbb{E}^{\mathbb{P}}[V(yY)]}{\varepsilon} = \mathbb{E}^{\mathbb{P}}[YV'(yY)].$$

Now consider the map $g(y) := \mathbb{E}^{\mathbb{P}}[YV'(yY)]$. We note that g is strictly increasing by the strict convexity of V. As $\lim_{y \to 0} yV'(y) = 0$, there is some constant $C > 0$ such that $YV'(yY) > -C$ for $y > 1$. By the

Inada conditions, it follows that

$$\lim_{y\to\infty} g(y) \geq \mathbb{E}^{\mathbb{P}}[\liminf_{y\to\infty} YV'(yY)] = \infty.$$

Denote by $\hat{y}$ the unique y such that $V'(\hat{y}) = 0$. We can split g into two parts:

$$\lim_{y\to 0} g(y) = \underbrace{\lim_{y\to 0} \mathbb{E}^{\mathbb{P}}[YV'(yY)\mathbf{1}_{yY\leq\hat{y}}]}_{1)} + \underbrace{\lim_{y\to 0} \mathbb{E}^{\mathbb{P}}[YV'(yY)\mathbf{1}_{yY>\hat{y}}]}_{2)}.$$

For the first part we get, as $\lim_{y\to 0} V'(y) = -\infty$

$$1) \leq \mathbb{E}^{\mathbb{P}}[\limsup_{y\to 0} YV'(yY)\mathbf{1}_{yY\leq\hat{y}}] = -\infty,$$

and for the second term, since $y < 1$, we have that $V'(yY) \leq V'(Y)$ and hence

$$2) = \lim_{y\to 0} \mathbb{E}^{\mathbb{P}}[YV'(Y)\mathbf{1}_{yY>\hat{y}}] \leq C\mathbb{E}^{\mathbb{P}}[V(Y)] < \infty.$$

Hence $g(y) \to -\infty$ for $y \to 0$ and together with the fact that $g(y)$ is strictly increasing we have completed the proof. □

In what follows, we present results that give some upper resp. lower bounds on $\inf_{y>0} \frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)] + u)$, which will be used later to get bounds for the average utility indifference price (more precise bounds on the entropic penalty functional) and to derive convergence results.

Lemma 4.1.3. ([Rob13, Lemma A.4]) *Let $\alpha > 0, p > 1$ and $l > 0$. Let furthermore $u > 0, Y \geq 0$ be such that $\mathbb{E}^{\mathbb{P}}[Y] = 1$. Then, for each $0 < \varepsilon < u$, there exists a constant $\overline{C}(\varepsilon, U) > 0$ (independent of Y and u) such that*

$$\inf_{y>0} \frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)] + u) \leq \overline{C}(\varepsilon, U) + \begin{cases} \dfrac{1+\varepsilon}{\alpha}\mathbb{E}^{\mathbb{P}}[Y\log(Y)] + u & \text{for } U \in \mathcal{U}_\alpha \\ \dfrac{1}{\alpha}\mathbb{E}^{\mathbb{P}}[Y\log(Y)] + u & \text{for } U \in \tilde{\mathcal{U}}_\alpha \\ (l(u+\varepsilon))^{\frac{1}{p}}\left((1+\varepsilon)\mathbb{E}^{\mathbb{P}}[Y^\gamma]\right)^{\frac{1}{\gamma}} & \text{for } U \in \mathcal{U}_{p,l}. \end{cases}$$

Proof. Let $U \in \mathcal{U}_\alpha$. We first note that $\inf_{y>0} \frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)] + u) \leq \mathbb{E}^{\mathbb{P}}[V(Y)] + u$. By the proof of Lemma 4.1.1, there is a constant $\overline{C} = \overline{C}(\varepsilon, U)$ such that

$$\mathbb{E}^{\mathbb{P}}[V(Y)] + u \leq \overline{C} + \frac{(1+\varepsilon)}{\alpha}\mathbb{E}^{\mathbb{P}}[Y\log(Y)] + u.$$

Hence the case for $U \in \mathcal{U}_\alpha$ is proven.
Consider $U \in \tilde{\mathcal{U}}_\alpha$ and define

$$f_U(z) := V(z) - \frac{1}{\alpha}z(\log z - 1) = V(z) - V_\alpha(z).$$

We then get that $\limsup_{z\to\infty} \frac{|f_U(z)|}{z} < \infty$ as $\lim_{z\to\infty} \frac{V(z)}{V_\alpha(z)} = 1$ and by the definition of $\tilde{\mathcal{U}}_\alpha$. Since $f_U(0) = 0$, there exists some $M = M(\varepsilon, U)$ such that for $z > 0$ we have $f_U(z) \leq M(1+z)$ (also here, we rely on the property of $U \in \tilde{\mathcal{U}}_\alpha$) and thus

$$\mathbb{E}^{\mathbb{P}}[V(Y)] + u \leq \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}[Y\log(Y)] + \mathbb{E}^{\mathbb{P}}[f_U(Y)] + u \leq \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}[Y\log(Y)] + 2M + u.$$

Lastly, let $U \in \mathcal{U}_{p,l}$. Recall the property of U in (2.0.2). For $0 < \varepsilon < u$, we then get that, as $\lim_{z\to 0} V(z) = 0$, there exists some $M = M(\varepsilon, U)$ such that $V(z) < \varepsilon$ for $z < \frac{1}{M}$, $V(z) \leq M$ for $\frac{1}{M} \leq z \leq M$ and $V(z) \leq (1+\varepsilon)\hat{l}z^{\gamma}$ for $z > M$. Thus conditioning on these three sets gives

$$\frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)] + u) \leq \frac{u+\varepsilon}{y} + M \underbrace{\frac{1}{y}\mathbb{P}[yY \geq \frac{1}{M}]}_{\leq M \text{ by Markov ineq.}} + (1+\varepsilon)\hat{l}y^{\gamma-1}\mathbb{E}^{\mathbb{P}}[Y^{\gamma}].$$

From this it follows that

$$\begin{aligned}\inf_{y>0}\left(\frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)] + u)\right) &\leq M^2(\varepsilon, U) + \inf_{y>0}\left(\frac{u+\varepsilon}{y} + (1+\varepsilon)\hat{l}\mathbb{E}^{\mathbb{P}}\left[Y^{\gamma}\right]\right) \\ &\leq M^2(\varepsilon, U) + \gamma\left(\frac{u+\varepsilon}{\gamma-1}\right)^{\frac{1}{p}}\left((1+\varepsilon)\hat{l}\mathbb{E}^{\mathbb{P}}[Y^{\gamma}]\right)^{\frac{1}{\gamma}} \\ &= \overline{C}(\varepsilon, U) + (l(u+\varepsilon))^{\frac{1}{p}}\left((1+\varepsilon)\mathbb{E}^{\mathbb{P}}[Y^{\gamma}]\right)^{\frac{1}{\gamma}},\end{aligned}$$

as desired, which completes the proof. □

Lemma 4.1.4. ([Rob13, Lemma A.5]) *Let $\alpha > 0, p > 1$ and $l > 0$. Let furthermore $u > 0, Y \geq 0$ be such that $\mathbb{E}^{\mathbb{P}}[Y] = 1$. Then for each $0 < \varepsilon < \min\{u, 1\}$, there exist constants $\underline{C}(\varepsilon, U)$ and $\underline{D}(\varepsilon, U) > 0$ (independent of Y and u) such that*

$$\inf_{y>0}\frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)] + u) \geq -\underline{C}(\varepsilon, U) + \begin{cases}\frac{1-\varepsilon}{\alpha}\mathbb{E}^{\mathbb{P}}[Y\log(Y)] + \underline{D}(\varepsilon, U)\log(u) & \text{for } U \in \mathcal{U}_{\alpha} \\ \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}[Y\log(Y)] + \underline{D}(\varepsilon, U)\log(u) & \text{for } U \in \tilde{\mathcal{U}}_{\alpha} \\ \left(l\left(u - \frac{\varepsilon}{2}\right)\right)^{\frac{1}{p}}\left((1-\varepsilon)\mathbb{E}^{\mathbb{P}}[Y^{\gamma}]\right)^{\frac{1}{\gamma}} & \text{for } U \in \mathcal{U}_{p,l}.\end{cases}$$

Proof. The proof follows the same pattern as the proof of Lemma 4.1.3.
Let $0 < \varepsilon < \min\{u, 1\}$. Let $U \in \mathcal{U}_{\alpha}$. We recall the properties of U and V respectively in (2.0.1) and that $V(0) = 0$. This gives us the existence of some $M = M(\varepsilon, U)$ such that on $\{z < \frac{1}{M}\}$ we have $V(z) \geq -\frac{\varepsilon}{2}$, on $\{\frac{1}{M} \leq z \leq M\}$ we have $V(z) \geq U(0)$ and finally $V(z) \geq (1-\varepsilon)\frac{1}{\alpha}z(\log z - 1)$ anywhere else.
Again, we condition on the three sets which leads us to, as $U(0) < 0$

$$\frac{1}{y}\left(\mathbb{E}^{\mathbb{P}}[V(yY)] + u\right) \geq \frac{u - \frac{\varepsilon}{2}}{y} + U(0) \underbrace{\frac{1}{y}\mathbb{P}[yY \geq \frac{1}{M}]}_{\leq M \text{ by Markov ineq.}} + \frac{1-\varepsilon}{\alpha}\left(\mathbb{E}^{\mathbb{P}}[Y(\log(yY) - 1)(\mathbf{1} - \mathbf{1}_{yY \leq M})]\right).$$

Now we use that $\mathbb{E}^{\mathbb{P}}[Y(\log(yY) - 1)\mathbf{1}_{yY \leq M}] \leq \log(M) - 1$ which yields

$$\frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)] + u) \geq \frac{u - \frac{\varepsilon}{2}}{y} + U(0)M + \frac{1-\varepsilon}{\alpha}\log(y) + \frac{1-\varepsilon}{\alpha}\mathbb{E}^{\mathbb{P}}[Y\log(Y)] - \frac{1-\varepsilon}{\alpha}\log(M).$$

We have that

$$U(0)M + \inf_{y>0}\left(\frac{1-\varepsilon}{\alpha}\log(y) + \frac{u - \frac{\varepsilon}{2}}{y}\right) = U(0)M + \frac{1-\varepsilon}{\alpha}\left(1 + \log\left(\frac{\alpha(u - \frac{\varepsilon}{2})}{1-\varepsilon}\right)\right). \tag{4.1.1}$$

Subtracting $\frac{1-\varepsilon}{\alpha}\log(M)$ on both sides of (4.1.1) and adding and subtracting $\frac{1-\varepsilon}{\alpha}\log(u)$ on the right-hand side gives

$$\inf_{y\geq 0}\frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)]+u)\geq U(0)M+\frac{1-\varepsilon}{\alpha}\left(1+\log\left(\frac{\alpha(u-\frac{\varepsilon}{2})}{1-\varepsilon}\right)\right)-\frac{1-\varepsilon}{\alpha}\log(u)-\frac{1-\varepsilon}{\alpha}\log(M)$$
$$+\frac{1-\varepsilon}{\alpha}\mathbb{E}^{\mathbb{P}}\left[Y\log(Y)\right]+\frac{1-\varepsilon}{\alpha}\log(u).$$

Simplifying all the terms gives us the conditions

$$-\underline{C}(\varepsilon,U):=U(0)M+\frac{1-\varepsilon}{\alpha}\left(1+\log\left(\frac{\alpha}{2M(1-\varepsilon)}\right)\right)$$
$$\underline{D}(\varepsilon,U):=\frac{1-\varepsilon}{\alpha},$$

where we used that $1-\frac{\varepsilon}{2u}\geq\frac{1}{2}$.
Let $U\in\tilde{\mathcal{U}}_\alpha$ and define $f_U(z)$ again by

$$f_U(z):=V(z)-\frac{1}{\alpha}z(\log(z)-1).$$

From the proof of Lemma 4.1.3, we know that $f(0)=0$ and that $\limsup_{z\to\infty}\frac{|f_U(z)|}{z}<\infty$. For $\varepsilon>0$, there exist $M=M(\varepsilon,U)$ and $K=K(\varepsilon,U)$ such that on $\{z<\frac{1}{M}\}$, we have $f_U(z)\geq-\frac{\varepsilon}{2}$, on $\{\frac{1}{M}\leq z\leq M\}$, we have $f_U(z)\geq -K$ and finally $f_U(z)\geq -Kz$ anywhere else. Therefore, again by conditioning on the three sets, we get

$$\frac{1}{y}\mathbb{E}^{\mathbb{P}}[f_U(yY)]\geq-\frac{\varepsilon}{2y}-K\underbrace{\frac{1}{y}\mathbb{P}[yY\geq\frac{1}{M}]}_{\leq M\text{ by Markov ineq.}}-K\mathbb{E}^{\mathbb{P}}[Y\mathbf{1}_{yY>M}]\geq-\frac{\varepsilon}{2y}-K(1+M).$$

This leads to

$$\frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)]+u)=\frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}[Y(\log(Y)-1)]+\frac{1}{y}\mathbb{E}^{\mathbb{P}}[f_U(yY)]+\frac{u}{y}$$
$$\geq\frac{u-\frac{\varepsilon}{2}}{y}+\frac{1}{\alpha}\log(y)+\frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}[Y\log(Y)]-\frac{1}{\alpha}-K(1+M).$$

Calculations as in the first part of the proof yield the desired result.
Finally, for $U\in\mathcal{U}_{p,l}$, as seen in the proof of Lemma 4.1.3, the estimates on the three different sets are the same, except for $z>M$, we have $V(z)\geq(1-\varepsilon)\hat{l}z^\gamma$ which then gives

$$\frac{1}{y}(\mathbb{E}^{\mathbb{P}}[V(yY)]+u)\geq\frac{u-\frac{\varepsilon}{2}}{y}+U(0)M+\underbrace{(1-\varepsilon)\hat{l}y^{\gamma-1}\mathbb{E}^{\mathbb{P}}[Y^\gamma(1-\mathbf{1}_{yY\leq M})]}_{\leq(1-\varepsilon)\hat{l}(y^{\gamma-1}\mathbb{E}^{\mathbb{P}}[Y^\gamma]-M^{\gamma-1})}.$$

Again, calculations as in the first part of the proof yield the desired result. □

Now we have finally established results that will be of great help in the following study of price convergence.

4.2 Convergence of Prices in the Large Claim Limit

To study the convergence of prices, we assume that we work on a sequence of filtered probability spaces $(\Omega^n,(\mathcal{F}_t^n)_{0\leq t\leq T},\mathbb{F}^n,\mathbb{P}^n)$ satisfying the usual conditions.

4.2.1 Indifference Prices in the Large Claim Limit for Exponential Utility Functions

The goal is now to study the behavior of the average utility indifference price $p_U^n(x, q_n; h^n)$, in particular the limit of it as $q_n \to \infty$ and $\varrho_n \to 1$. **Large Claim Limit** indicates on the one hand that the position size grows to infinity and on the other hand that, asymptotically, the markets $(\Omega^n, (\mathcal{F}_t^n)_{0 \leq t \leq T}, \mathbb{F}^n, \mathbb{P}^n)$ converge to a complete market as hedging errors become negligible while not permitting any arbitrage opportunity. Therefore, we are interested in the convergence of the markets *and* the position sizes simultaneously. To ensure no asymptotic arbitrage (no nirvana), some additional assumptions will be needed. We are already aware of the conditions for excluding arbitrage opportunities even in the limit of this sequence of markets: see Section 2.2.1, in particular Lemma 2.2.1.

4.2.1.1 Convergence of Prices

These are the afore-remarked assumptions:

Assumption 2. Uniform boundedness of h, i.e. $\|h\| := \sup_n \|h^n\|_{L^\infty(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)} < \infty$.

Assumption 3. $\tilde{\mathcal{M}}^n \neq \emptyset$ for each n and $\limsup_{n \to \infty} \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} H(\mathbb{Q}^n | \mathbb{P}^n) < \infty$.

Note. It is Assumption 3 that rules out arbitrage opportunities for every market $(\Omega^n, (\mathcal{F}_t^n)_{0 \leq t \leq T}, \mathbb{F}^n, \mathbb{P}^n)$ on $[0, T]$ when investing in S^n as well as in the limit when $n \to \infty$. We have seen this in Lemma 2.2.1. [OŽ09, Theorem 1.9]

The next theorem shows that under these assumptions, the difference between two average utility indifference prices vanishes for utilities belonging to the same (exponential) class.

Theorem 4.2.1. ([Rob13, Theorem 3.4]) *Let $\alpha > 0$ and Assumption 2 and Assumption 3 hold. If $q_n \to \infty$, then for all $U_1, U_2 \in \mathcal{U}_\alpha$ and $x_1, x_2 \in \mathbb{R}$, we have*

$$\lim_{n \to \infty} |p_{U_1}^n(x_1, q_n; h^n) - p_{U_2}^n(x_2, q_n; h^n)| = 0.$$

Remark 4.2.1.

- As we have that for $U \in \mathcal{U}_\alpha : p_U^n(x, q_n; h^n) = -p_U^n(x, -q_n; -h^n)$, above result can also be stated for $q_n \to -\infty$, i.e. for large short positions. In the sequel we will just speak of 'large positions' and mean either cases.

- By letting n tend to ∞ or roughly speaking in the limit market, the individual utility function $U \in \mathcal{U}_\alpha$ has no impact and $p_U^n(x, q_n; h^n)$ converges to $p_{U_\alpha}^n(x, q_n; h^n)$ - to the price an investor with the canonical exponential utility function is willing to pay for large claims. Hence for pricing large claims, one could directly work with the canonical example of $U_\alpha \in \mathcal{U}_\alpha$.

- The economical interpretation of this is the following: An investor should identify her rate of decay for large negative wealths and if she finds that $\lim_{x \to -\infty} -\frac{1}{x} \log(-U(x)) = \bar{\alpha}$, then she should price as she were having utility $U_{\bar{\alpha}} = -\frac{1}{\bar{\alpha}} e^{-\bar{\alpha} x}$.

- In summary, above theorem states that only the asymptotic attitude towards big losses will have an impact on prices. Due to the fact that exponential utility prices are independent of the current wealth, this is plausible.

Proof. Let $\alpha > 0, U \in \mathcal{U}_\alpha$ and $x \in \mathbb{R}$. By Lemma 2.2.1, we can choose $\varepsilon > 0$ such that for n large enough

$$\varepsilon < -u_U^n(x, q; 0).$$

We rely on the following lemma:

Lemma 4.2.1. ([OŽ09, Proposition 7.2 (vi)]) *Assume that h^n is sub- and superreplicable (in the sense of (2.1.3)) and that the utility function $U \in C^2(\mathbb{R})$ satisfies the Inada conditions as well as the Conditions of Reasonable Asymptotic Elasticity. Then we have for V being the convex conjugate to U that*

$$p_U^n(x, q_n; h^n) = \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n(Y_T)\right] + \frac{1}{q_n} \alpha_U^n(\mathbb{Q}^n) \right), \tag{4.2.1}$$

where the entropic penalty functional α_U^n is given by

$$\alpha_U^n(\mathbb{Q}^n) := \inf_{y>0} \frac{1}{y} \left(\mathbb{E}^{\mathbb{P}^n}\left[V\left(y \frac{d\mathbb{Q}^n}{d\mathbb{P}^n} \right) \right] + xy - u_U^n(x, q_n; 0) \right). \tag{4.2.2}$$

Proof of Lemma 4.2.1. We refer to [OŽ09, Proposition 7.2]. □

Proof of Theorem 4.2.1 (continued). Write $Z^{\mathbb{Q}^n} := \frac{d\mathbb{Q}^n}{d\mathbb{P}^n}$ for the density of $\mathbb{Q}^n$ with respect to $\mathbb{P}^n$. Then, (4.2.1) becomes

$$p_U^n(x, q_n; h^n) = \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{P}^n}\left[h^n(Y_T) Z^{\mathbb{Q}^n} \right] + \frac{1}{q_n} \alpha_U^n(\mathbb{Q}^n) \right).$$

By Lemma 4.1.3 with $u = -u_U^n(x, q; 0)$ and $Y = Z^{\mathbb{Q}^n}$, we get that there is a constant $\overline{C} = \overline{C}(\varepsilon, U)$ such that

$$p_U^n(x, q_n; h^n) \leq \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n(Y_T)\right] + \frac{1+\varepsilon}{q_n \alpha} H(\mathbb{Q}^n | \mathbb{P}^n) \right) + \frac{x + \overline{C}(\varepsilon, U) - u_U^n(x, q_n; 0)}{q_n}.$$

Similarly from Lemma 4.1.4, we have

$$p_U^n(x, q_n; h^n) \geq \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \frac{1-\varepsilon}{q_n \alpha} H(\mathbb{Q}^n | \mathbb{P}^n) \right) + \frac{x - \underline{C}(\varepsilon, U) + \underline{D}(\varepsilon, U) \log(-u_U^n(x, q_n; 0))}{q_n}.$$

We then consider the function

$$f(\delta, n) := \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \delta H(\mathbb{Q}^n | \mathbb{P}^n) \right) \text{ for } \delta > 0,$$

which is obviously increasing in δ. Assumption 3 yields that

$$f(\delta, n) \leq \|h\| + K\delta \text{ for some } K > 0,$$

as $\inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} H(\mathbb{Q}^n | \mathbb{P}^n) \leq \limsup_{n\to\infty} \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} H(\mathbb{Q}^n | \mathbb{P}^n) =: K < \infty$.
For $0 < \delta < \gamma$ and for any $\mathbb{Q} \in \tilde{\mathcal{M}}^n$, we get the trivial inequality

$$\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \gamma H(\mathbb{Q}^n | \mathbb{P}^n) \leq \frac{\gamma}{\delta} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \delta H(\mathbb{Q}^n | \mathbb{P}^n) \right) + \left(\frac{\gamma}{\delta} - 1 \right) \|h\|,$$

hence

$$f(\gamma, n) - f(\delta, n) \leq \left(\frac{\gamma}{\delta} - 1 \right) (f(\delta, n) + \|h\|) \leq \left(\frac{\gamma}{\delta} - 1 \right) (2\|h\| + K\delta).$$

Let $U_1, U_2 \in \mathcal{U}_\alpha$ and $x_1, x_2 \in \mathbb{R}$. Then choose $\varepsilon > 0$ such that

$$\varepsilon \leq -u^n_{U_i}(x_i, q; 0) \leq -U_i(x_i) \text{ for } i = 1, 2.$$

We have that

$$\begin{aligned} &p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n) \\ &\qquad \leq \frac{(x_1 + \underline{C}(\varepsilon, U_1) - u^n_{U_1}(x_1, q_n; 0)) - (x_2 - \underline{C}(\varepsilon, U_2) + \underline{D}(\varepsilon, U_2)\log(-u^n_{U_2}(x_2, q_n; 0)))}{q_n} \\ &\qquad + f\left(\frac{1+\varepsilon}{q_n\alpha}, n\right) - f\left(\frac{1-\varepsilon}{q_n\alpha}, n\right) \\ &\qquad \leq \frac{C(n,\varepsilon)}{q_n} + f\left(\frac{1+\varepsilon}{q_n\alpha}, n\right) - f\left(\frac{1-\varepsilon}{q_n\alpha}, n\right) \\ &\qquad \leq \frac{C(n,\varepsilon)}{q_n} + \left(\frac{1-\varepsilon}{1+\varepsilon} - 1\right)\left(2\|h\| + K\frac{1-\varepsilon}{q_n\alpha}\right), \end{aligned}$$

for $C(n, \varepsilon)$ such that $\frac{C(n,\varepsilon)}{q_n} \to 0$ as $q_n \to \infty$.
Thus

$$\limsup_{n\to\infty} \left(p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n)\right) \leq 2\|h\| \left(\frac{1-\varepsilon}{1+\varepsilon} - 1\right). \tag{4.2.3}$$

As the left-hand side of (4.2.3) is independent of ε, we can pass to the limit $\varepsilon \to 0$ and get the desired result, as we can interchange the role of U_1, U_2 and x_1, x_2. □

4.2.1.2 Convergence of Total Quantities

We now know that the difference between two average utility indifference prices vanishes in the limit for utility functions belonging to the same exponential class. A natural question that one might be interested in is whether the total money difference remains finite. Put differently, we investigate whether the speed of price convergence is at least linear, that is, whether we have

$$q_n \left|p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n)\right| < \infty, \text{ as } n \to \infty\ ?$$

The answer is indeed yes, but not anymore for any $U \in \mathcal{U}_\alpha$ but rather for $U \in \tilde{\mathcal{U}}_\alpha \subset \mathcal{U}_\alpha$. This includes the additional requirement to $U \in \mathcal{U}_\alpha$ of

$$0 < \liminf_{n\to-\infty} \frac{U(x)}{U_\alpha(x)} \leq \limsup_{n\to-\infty} \frac{U(x)}{U_\alpha(x)} < \infty. \tag{4.2.4}$$

We have seen the example of $U(x) \in \mathcal{U}_\alpha \setminus \tilde{\mathcal{U}}_\alpha$ given by $U(x) = -\frac{1}{x}U_\alpha(x)$ for large negative wealths. [Rob13, Example 3.12] shows that for such a utility function, the total money error does not remain bounded.
Having strengthened the assumption to U, it then turns out that the assumption to h^n can be weakened. For the rest of this paragraph, we do not anymore assume the uniform boundedness of h^n, but rather:

Assumption 4. For each n, we assume that $h^n \in L^\infty(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$.

Theorem 4.2.2. ([Rob13, Theorem 3.9]) *Let $\alpha > 0$. Assume further that Assumption 3 and Assumption*

4 hold. If $q_n \to \infty$, then for all $U_1, U_2 \in \tilde{\mathcal{U}}_\alpha$ and $x_1, x_2 \in \mathbb{R}$, we have

$$\limsup_{n\to\infty} q_n \left| p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n) \right| < \infty.$$

Note. This theorem essentially states that an investor with utility function $U \in \tilde{\mathcal{U}}_\alpha$ may price as if she were an investor with exponential utility function $U_\alpha \in \tilde{\mathcal{U}}_\alpha$ and the resulting error of money spent remains bounded - but the latter holding only under stronger assumptions to the utility function.
From a purely mathematical point of view, the requirement on U belonging to $\tilde{\mathcal{U}}_\alpha$ is needed as we then get from Lemma 4.1.3 and Lemma 4.1.4 tighter bounds which gives us that the resulting relative entropy terms are independent from ε, hence they will vanish in the difference.

Proof. The proof is similar to the proof of Theorem 4.2.1. We let $\alpha > 0$, $U \in \tilde{\mathcal{U}}_\alpha$ and $x \in \mathbb{R}$. From Lemma 2.2.1, it follows that there exists $\varepsilon > 0$ such that $-u^n_U(x, q_n; 0) \geq \varepsilon$ for n large enough.
Using the representation of $p^n_U(x, q_n; h^n)$ in terms of the entropic penalty functional $\alpha^n_U(\mathbb{Q}^n)$ in (4.2.2) and referring to Lemma 4.1.3, we get the following upper bound

$$p^n_U(x, q_n; h^n) \leq \frac{x + \overline{C}(\varepsilon, U) - u^n_U(x, q_n; 0)}{q_n} + \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \frac{1}{\alpha q_n} H(\mathbb{Q}^n|\mathbb{P}^n) \right).$$

Similarly, using Lemma 4.1.4, we get the lower bound

$$p^n_U(x, q_n; h^n) \geq \frac{x - \underline{C}(\varepsilon, U) + \underline{D}(\varepsilon, U)\log(-u^n_U(x, q_n; 0))}{q_n} + \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \frac{1}{\alpha q_n} H(\mathbb{Q}^n|\mathbb{P}^n) \right).$$

Let $U_1, U_2 \in \tilde{\mathcal{U}}_\alpha$ and $x_1, x_2 \in \mathbb{R}$. Again, we choose $\varepsilon > 0$ such that $\varepsilon \leq -u^n_{U_i}(x_i, q_n; 0) \leq -U_i(x_i)$ for $i = 1, 2$. Then

$$\begin{aligned} &q_n \left| p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n) \right| \\ &\leq \left| (x_1 + \overline{C}(\varepsilon, U) - u^n_U(x_1, q_n; 0)) - (x_2 - \underline{C}(\varepsilon, U) + \underline{D}(\varepsilon, U)\log(-u^n_U(x_2, q_n; 0)) \right| =: C(\varepsilon, U). \end{aligned}$$

We note that $\sup_n C(\varepsilon, U) < \infty$, hence we can take the limit for $n \to \infty$ and interchange the roles of U_1, U_2 and x_1, x_2 to get the desired result. ☐

4.2.1.3 Pricing in a Fixed Market when only the Position is Changing

In what follows, we investigate the behavior of $p_U(x, q_n; h)$ as $n \to \infty$, i.e. the behavior of the average utility indifference price in a constant kept market where the only change occurs in the position size q_n of the (constant kept) claim $h^n \equiv h$. We have given heuristic observations in the beginning of this chapter which are now strengthened by mathematical results.

Theorem 4.2.3. ([Rob13, (3.7)]) *Under Assumption 2 and Assumption 3, we have that for all $U \in \mathcal{U}_\alpha$*

$$\lim_{n\to\infty} p_U(x, q_n; h) = \inf_{\mathbb{Q} \in \tilde{\mathcal{M}}} \mathbb{E}^{\mathbb{Q}}[h(Y_T)].$$

Proof. This proof can be found in [OŽ09, Proposition 7.5 (ii)]. ☐

Remark 4.2.2. The price an investor with utility function $U \in \mathcal{U}_\alpha$ is willing to pay per unit for an incredibly large position in a constant kept market is given by the superreplication price (see Theorem 2.2.1). Here we see the **large position effect** for the very first time. By purchasing enormously large positions, the investor is able to push prices towards the minimal possible arbitrage-free price. This is reasonable as by the non-vanishing hedging error, the investor is exposed to huge risk when holding large positions. Therefore, in the limit, she is only willing to pay the lowest arbitrage-free price.

4.2.1.4 Interchangeability of α and q in $p^n_{U_\alpha}(x, q; h^n)$

In this paragraph, we want to emphasize the interchangeability with respect to indifference pricing of the absolute risk aversion parameter α and the number of units held in the claim for the case of an exponential utility $U_\alpha \in \mathcal{U}_\alpha$. More concretely, recalling the explicit representation in (3.2.11), we have:

$$p^n_{U_\alpha}(x, q; h^n) = \frac{1}{\alpha q} \log\left(\frac{u^n_{U_\alpha}(0, q; 0)}{u^n_{U_\alpha}(0, q; h^n)}\right) = \frac{1}{\alpha q} \log\left(\frac{q u^n_{U_{q\alpha}}(0, 1; 0)}{q u^n_{U_{q\alpha}}(0, 1; h^n)}\right) = p^n_{U_{q\alpha}}(x, 1; h^n), \tag{4.2.5}$$

as for any $q, \alpha > 0$, we have that $U_\alpha(qx) = qU_{q\alpha}(x)$.
The left-hand side of (4.2.5) is the price, which an α-risk averse investor would pay per unit of h^n to be indifferent between owning q units of h^n or not while the right-hand side is the price an αq-risk averse investor would pay per unit to be indifferent between owning 1 unit of h^n or not.
Giving an illustration, assume that $q = 2$. Hence the price per unit such that an investor being indifferent between holding 2 units of h^n and holding none coincides with the price per unit for which a twice as high risk averse investor would be indifferent between holding one unit of the claim and holding none.
If we assume that q varies with n and $q_n \to \infty$, but the market and the claim are kept constant, then above result can be written as

$$p_{U_\alpha}(x, q_n; h) = p_{U_{q_n\alpha}}(x, 1; h).$$

Taking limits as $n \to \infty$, we get by Theorem 4.2.3

$$\lim_{n\to\infty} p_{U_{q_n\alpha}}(x, 1; h) = \inf_{\mathbb{Q}\in\tilde{\mathcal{M}}} \mathbb{E}^{\mathbb{Q}}[h(Y_T)].$$

The message of this is that as the absolute risk aversion $q_n\alpha$ increases to infinity, and the markets and claims are kept constant, we have that the average utility indifference price per unit of the claim h converges to the minimal arbitrage-free price (= superreplication price). Infinite risk aversion is hard to imagine - a good way is to think of it as 'comparing with the worst case scenario'. By this, it obviously makes sense that superhedging goes together with infinite risk aversion.
We mentioned earlier in (3.2.22) that the limit of the price as risk aversion α goes to zero coincides with the marginal utility price which is given by the arbitrage-free price. However, here we have established the limiting prices for risk aversion going to plus infinity.

4.2.2 Convergence of Prices for Utilities with a Power-like Decay

The goal is now to establish similar results as in Theorem 4.2.1 for utility functions with a power-like decay for large negative wealths. In order to obtain such results, one has to adjust the rate at which q_n becomes large in a suitable way. The reason for modifying the rate can best be observed in the proof of the following theorem. We show that it does not work for $q_n \to \infty$, but for $k_n q_n \to \infty$ it does, for some k_n.
The equivalent to Assumption 3 for the case of power law utility to ensure no nirvana is the following.

Assumption 5. Assume that for each n: $\hat{\mathcal{M}}^n_V \neq \emptyset$ and that $\limsup_{n\to\infty} \inf_{\mathbb{Q}^n\in\hat{\mathcal{M}}^n} \mathbb{E}^{\mathbb{P}^n}[(\frac{d\mathbb{Q}^n}{d\mathbb{P}^n})^\gamma] < \infty$.

Theorem 4.2.4. ([Rob13, Proposition 4.3]) *Let $p > 1$ and $l > 0$. Let Assumption 2 and Assumption 5 hold. If $q_n \to \infty$, then for all $U_1, U_2 \in \mathcal{U}_{p,l}$ and $x_1, x_2 \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} \left| p^n_{U_1}(x_1, q_n(-u^n_{U_1}(x_1, q_n; 0))^{\frac{1}{p}}; h^n) - p^n_{U_2}(x_2, q_n(-u^n_{U_2}(x_2, q_n; 0))^{\frac{1}{p}}; h^n) \right| = 0.$$

Proof. The proof is again similar to the one of Theorem 4.2.1. Let $p > 1, l > 0, U \in \mathcal{U}_{p,l}$ and $x \in \mathbb{R}$. We recall Lemma 4.2.1, which gives us a pricing formula in terms of the entropic penalty functional, which is valid for any utility U on the real line, therefore also for $U \in \mathcal{U}_{p,l}$. Hence

$$p^n_U(x, q_n; h^n) = \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n_V} \left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n(Y_T)\right] + \frac{1}{q_n}\alpha^n_U(\mathbb{Q}^n) \right),$$

for

$$\alpha^n_U(\mathbb{Q}^n) = \inf_{y>0} \frac{1}{y} \left(\mathbb{E}^{\mathbb{P}^n}\left[V\left(y\frac{d\mathbb{Q}^n}{d\mathbb{P}^n} \right) \right] + xy - u^n_U(x, q_n; 0) \right).$$

Then by Lemma 2.2.1, for $\varepsilon > 0$, we have that $\varepsilon < -u^n_U(x, q_n; 0)$ for n sufficiently large. Lemma 4.1.3 implies then that for an arbitrary position size q_n, we have

$$p^n_U(x, q_n; h^n) \leq \frac{x + \overline{C}(\varepsilon, U)}{q_n} + \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n_V} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \frac{1}{q_n}\left(l(-u^n_U(x, q_n; 0) + \varepsilon)\right)^{\frac{1}{p}} (1+\varepsilon)^{\frac{1}{\gamma}} \mathbb{E}^{\mathbb{P}^n}[(Z^{\mathbb{Q}^n})^\gamma]^{\frac{1}{\gamma}} \right).$$

In the same way, by Lemma 4.1.4, we have

$$p^n_U(x, q_n; h^n) \geq \frac{x - \underline{C}(\varepsilon, U)}{q_n} + \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n_V} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \frac{1}{q_n}\left(l(-u^n_U(x, q_n; 0) - \frac{\varepsilon}{2})\right)^{\frac{1}{p}} (1-\varepsilon)^{\frac{1}{\gamma}} \mathbb{E}^{\mathbb{P}^n}[(Z^{\mathbb{Q}^n})^\gamma]^{\frac{1}{\gamma}} \right).$$

Consider the function

$$\hat{f}(\delta, n) := \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n_V} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \delta \mathbb{E}^{\mathbb{P}^n}[(Z^{\mathbb{Q}^n})^\gamma]^{\frac{1}{\gamma}} \right) \text{ for } \delta > 0,$$

which is obviously increasing in δ and using Assumption 5, we get that $\hat{f}(\delta, n) \leq \|h\| + K\delta$.
Let $0 < \delta < \gamma$. Then for $\mathbb{Q}^n \in \hat{\mathcal{M}}^n_V$, we get the trivial inequality

$$\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \gamma H(\mathbb{Q}^n|\mathbb{P}^n) \leq \frac{\gamma}{\delta}\left(\mathbb{E}^{\mathbb{Q}^n}[h^n(Y_T)] + \delta \mathbb{E}^{\mathbb{P}^n}[(Z^{\mathbb{Q}^n})^\gamma]^{\frac{1}{\gamma}} \right) + \left(\frac{\gamma}{\delta} - 1 \right)\|h\|,$$

and hence

$$\hat{f}(\gamma, n) - \hat{f}(\delta, n) \leq \left(\frac{\gamma}{\delta} - 1 \right)\left(\hat{f}(\delta, n) + \|h\| \right) \leq \left(\frac{\gamma}{\delta} - 1 \right)(K\delta + 2\|h\|).$$

As there is no asymptotic arbitrage, i.e. $\limsup_{n\to\infty} u^n_U(x, q_n; 0) < U(\infty) = 0$ and since $u^n_U(x, q_n; 0) \geq U(x)$, there exists some constant $M > 0$ such that $\frac{1}{M} \leq -u^n_U(x, q_n; 0) \leq M$ for n sufficiently large.
Now we proceed with the proof for the position size equal to q_n and see that the rate at which the position size increases has to be adjusted to get the desired result.
Let $U_1, U_2 \in \mathcal{U}_{p,l}$ and $x_1, x_2 \in \mathbb{R}$. Then

$$p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n) \leq \frac{x_1 + \overline{C}(\varepsilon, U_1) - x_2 + \underline{C}(\varepsilon, U_2)}{q_n} + \hat{f}\left(\frac{1}{q_n}\delta^+(\varepsilon, n), n \right) - \hat{f}\left(\frac{1}{q_n}\delta^-(\varepsilon, n), n \right),$$

for

$$\delta^+(\varepsilon,n) := \left(l(-u^n_{U_1}(x_1,q_n;0)+\varepsilon)\right)^{\frac{1}{p}}(1+\varepsilon)^{\frac{1}{\gamma}}$$
$$\delta^-(\varepsilon,n) := \left(l(-u^n_{U_2}(x_2,q_n;0)-\frac{\varepsilon}{2})\right)^{\frac{1}{p}}(1-\varepsilon)^{\frac{1}{\gamma}}.$$

The first term in above inequality vanishes as $q_n\to\infty$ hence can be disregarded. For the difference of the two $\hat{f}(\delta,n)$ terms, we get by above observations that

$$\frac{\delta^+(\varepsilon,n)}{\delta^-(\varepsilon,n)} \leq \frac{(-M-\varepsilon)^{\frac{1}{p}}(1+\varepsilon)^{\frac{1}{\gamma}}}{\left(-\frac{\varepsilon}{2}+\frac{1}{M}\right)^{\frac{1}{p}}(1-\varepsilon)^{\frac{1}{\gamma}}} \to (-M^2)^{\frac{1}{p}} < 0,$$

as $\varepsilon\to 0$. Moreover we have that

$$\frac{\delta^-(\varepsilon,n)}{q_n}\to 0,$$

as $n\to\infty$. It then follows that

$$\limsup_{n\to\infty}\left(p^n_{U_1}(x_1,q_n;h^n)-p^n_{U_2}(x_2,q_n;h^n)\right) \leq 2\|h\|\left(\frac{(-M-\varepsilon)^{\frac{1}{p}}(1+\varepsilon)^{\frac{1}{\gamma}}}{\left(\frac{\varepsilon}{2}-\frac{1}{M}\right)^{\frac{1}{p}}(1-\varepsilon)^{\frac{1}{\gamma}}}-1\right),$$

which does *not* converge to zero as $\varepsilon\to 0$.

Notwithstanding, under the adjusted rate $q_n(-u^n_{U_i}(x_i,q_n;0))^{\frac{1}{p}}$ (hence $k_n=(-u^n_{U_i}(x_i,q_n;0))^{\frac{1}{p}}$) at which the position size increases, we get by the very same calculations as above that

$$\begin{aligned}
&p^n_{U_1}(x_1,q_n(-u^n_{U_1}(x_1,q_n;0))^{\frac{1}{p}};h^n)-p^n_{U_2}(x_2,q_n(-u^n_{U_2}(x_2,q_n;0))^{\frac{1}{p}};h^n)\\
&\leq \frac{1}{q_n}\underbrace{\left(\frac{x_1+\overline{C}(\varepsilon,U_1)}{(-u^n_{U_1}(x_1,q_n;h))^{\frac{1}{p}}}-\frac{x_2+\underline{C}(\varepsilon,U_2)}{(-u^n_{U_2}(x_2,q_n;h))^{\frac{1}{p}}}\right)}_{=:C(\varepsilon,n)}+\hat{f}\left(\frac{1}{q_n}\delta^+(\varepsilon,n),n\right)-\hat{f}\left(\frac{1}{q_n}\delta^-(\varepsilon,n),n\right),
\end{aligned}$$

where

$$\delta^+(\varepsilon,n) := \left(1-\frac{\varepsilon}{(-u^n_{U_1}(x_1,q_n;0))}\right)^{\frac{1}{p}} l^{\frac{1}{p}}(1+\varepsilon)^{\frac{1}{\gamma}}$$
$$\delta^-(\varepsilon,n) := \left(1+\frac{\frac{\varepsilon}{2}}{u^n_{U_2}(x_2,q_n;0)}\right)^{\frac{1}{p}} l^{\frac{1}{p}}(1+\varepsilon)^{\frac{1}{\gamma}}.$$

We then get that, as $\varepsilon\to 0$,

$$\frac{\delta^+(\varepsilon,n)}{\delta^-(\varepsilon,n)} \leq \frac{(1+\varepsilon M)^{\frac{1}{p}}(1+\varepsilon)^{\frac{1}{\gamma}}}{\left(1-\frac{\varepsilon}{2}M\right)^{\frac{1}{p}}(1-\varepsilon)^{\frac{1}{\gamma}}} \to 1,$$

as desired and by copying above arguments, we are done. As $C(\varepsilon,n)q_n^{-1}\to 0$ for $n\to\infty$, this term has no impact on the limit. Therefore, we finally have that

$$\limsup_{n\to\infty}\left(p^n_{U_1}(x_1,q_n(-u^n_{U_1}(x_1))^{\frac{1}{p}};h^n)-p^n_{U_2}(x_2,q_n(-u^n_{U_2}(x_2))^{\frac{1}{p}};h^n)\right) \leq 2\|h\|\left(\frac{(1+\varepsilon M)^{\frac{1}{p}}(1+\varepsilon)^{\frac{1}{\gamma}}}{(1\varepsilon/2M)^{\frac{1}{p}}(1-\varepsilon)^{\frac{1}{\gamma}}}\right),$$

where we used that, for $n \to \infty$,

$$\frac{\delta^-(\varepsilon, n)}{q_n} \to 0.$$

As the right-hand side is independent of n, we can directly pass to the limit $\varepsilon \to 0$ and get the desired result by interchanging the roles of U_1, U_2 and x_1, x_2. $\square$

4.3 Pricing in a Stochastic Factor resp. Basis Risk Model

In this section, we present the aforementioned general stochastic factor model and get then the basis risk model as treated in Chapter 3 as a special case.

4.3.1 Model and Assumptions

We consider the following stochastic factor model, where the assets S_t^n and volatility Y_t satisfy the following stochastic differential equations (SDE):

$$\begin{aligned}\frac{dS_t^n}{S_t^n} &= \mu(Y_t)dt + \sigma(Y_t)\left(\varrho_n dW_t + \sqrt{1-\varrho_n^2}dB_t\right), S_0^n = 1 \\ dY(t) &= \nu(Y_t)dt + \eta(Y_t)dW_t,\end{aligned}$$

where W_t, B_t are independent Brownian motions, $\varrho_n \in [-1, 1]$ and $S_0^n = 1$. Moreover, we assume that $h^n = h(Y_T)$. Accordingly, each market consists of two risky assets having correlation ϱ_n, which is the only thing that changes with n.
Furthermore, we assume that each of our probability spaces $(\Omega^n, (\mathcal{F}_t^n)_{0\le t\le T}, \mathbb{F}^n, \mathbb{P}^n)$ is a two-dimensional Wiener space, where the filtration $\mathbb{F}^n = (\mathcal{F}_t^n)_{0\le t\le T}$ is the $\mathbb{P}^n$-augmented version of the right-continuous enlargement of the natural filtration $\mathcal{F}^{n,W,B}$ generated by W_t and B_t.
Moreover, for $\varrho_n = 1$, we denote by $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{F}, \mathbb{P})$ the asymptotically complete limiting market, for which we assume that it satisfies above conditions as well.
It will be crucial for our study to rely the following assumption:

Assumption 6. For $-\infty \le l < u \le \infty$, we set $E := (l, u)$[16]. Moreover, we assume $\nu, \eta : E \to \mathbb{R}$ are continuous functions and that $\eta^2(y) > 0$ for $y \in E$. Furthermore, we assume that the above SDE for Y_t admits a strong solution with respect to the $\mathbb{P}$-augmented filtration of W_t with $\mathbb{P}[Y_t \in E, 0 \le t \le T] = 1$. Moreover $\mu, \sigma : E \to \mathbb{R}$ are assumed to be measurable such that for $y \in E : \sigma^2(y) > 0$.
We set the Sharpe ratio

$$\lambda(y) := \frac{\mu(y)}{\sigma(y)}$$

and assume that $\lambda(y)$ is a bounded function on E.
Lastly, we assume that $h : E \mapsto \mathbb{R}$ is a bounded and continuous function and $\varrho_n \in (-1, 1)$.

Remark 4.3.1. The afore-introduced stochastic factor model is fairly general and by setting $E = (0, \infty), \mu(y) = \mu, \sigma(y) = \sigma, \nu(y) = \nu y, \eta(y) = \eta y$ for some $\mu, \nu \in \mathbb{R}$ and $\sigma, \eta \in \mathbb{R}^+$, which gives then that the traded asset S_t^n and the nontraded asset Y_t follow each a geometric Brownian motion with correlation ϱ_n, we obtain the basis risk model as studied in Chapter 3.

[16] Usually, the interval E is given by $E = \mathbb{R}$ or $E = \mathbb{R}^+$.

Note. Note that Assumption 6 implies Assumption 2 (obvious), 3 and 4. Concerning the implication of Assumption 3 and Assumption 4, we consider the following measure $\hat{\mathbb{Q}}^n$ given by

$$\frac{d\hat{\mathbb{Q}}^n}{d\mathbb{P}^n} := \mathcal{E}\left(-\int_0^{\cdot} \varrho_n \lambda(Y_t) dW_t - \int_0^{\cdot} \sqrt{1-\varrho_n^2}\lambda(Y_t) dB_t\right)_T.$$

Then under $\hat{\mathbb{Q}}^n$, the following processes are two independent Brownian motions

$$d\hat{W}_t^n := dW_t + \varrho_n \lambda(Y_t) dt, \qquad d\hat{B}_t^n := dB_t + \sqrt{1-\varrho_n^2}\lambda(Y_t) dt,$$

which gives

$$\begin{aligned} \frac{dS_t^n}{S_t^n} &= \mu(Y_t)dt + \sigma(Y_t)\left(\varrho_n\left(d\hat{W}_t^n - \varrho_n\lambda(Y_t)dt\right) + \sqrt{1-\varrho_n^2}\left(d\hat{B}_t^n - \sqrt{1-\varrho_n^2}\lambda(Y_t)dt\right)\right) \\ &= \varrho_n d\hat{W}_t^n + \sqrt{1-\varrho_n^2} d\hat{B}_t^n. \end{aligned}$$

Hence it follows that $\hat{\mathbb{Q}}^n \in \tilde{\mathcal{M}}^n$.
For completeness, we record here the $\hat{\mathbb{Q}}^n$-dynamics of Y_t, that is

$$dY_t = \left(\nu(Y_t) - \varrho_n \frac{\eta(Y_t)\mu(Y_t)}{\sigma(Y_t)}\right) dt + \eta(Y_t) d\hat{W}_t^n,$$

where we denote by $\delta(Y_t)$ its drift.
Moreover

$$\limsup_{n\to\infty} H(\hat{\mathbb{Q}}^n|\mathbb{P}^n) < \infty,$$

as $\lambda(Y_t)$ and ϱ_n are bounded on E and as the time horizon T is finite. Hence Assumption 3 is satisfied. For Assumption 4, note that for fixed $\gamma = \frac{p}{p-1} > 1$ with $p > 1$, we have

$$\mathbb{E}^{\mathbb{P}^n}\left[\left(\frac{d\hat{\mathbb{Q}}^n}{d\mathbb{P}^n}\right)^{\gamma}\right] \le \exp\left(\gamma T \sup_{y\in E} \lambda^2(y)\right) < \infty,$$

due to the same reasons as above. Hence Assumption 4 is satisfied, too and a no nirvana discussion is redundant.

Proposition 4.3.1. ([Teh04, Proposition 3.3]) *Under Assumption 6, we have that the value function $u^n_{U_\alpha}(x, q_n; h)$ for the canonical exponential utility $U_\alpha \in \mathcal{U}_\alpha$ admits the representation*

$$u^n_{U_\alpha}(x, q_n; h) = -\frac{1}{\alpha}\exp(-\alpha x)\mathbb{E}^{\mathbb{P}^n}\left[Z(\varrho_n)\exp\left(-(1-\varrho_n^2)\left(\alpha q_n h(Y_T) + \frac{1}{2}\int_0^T \lambda(Y_t)^2 dt\right)\right)\right]^{\frac{1}{1-\varrho_n^2}},$$

for

$$Z(\varrho_n) := \mathcal{E}\left(-\varrho_n \int_0^{\cdot} \lambda(Y_t) dW_t\right)_T; \; Z := Z(1)$$

being the projection of $\hat{\mathbb{Q}}^n$ on the filtration generated by W_t.

Proof. For a detailed proof, we refer to [Teh04, Proposition 3.3]. □

Note. Clearly, Z has all exponential moments, as $\lambda(y)$ is assumed to be bounded and T being finite.

Moreover

$$Z(\varrho_n)\exp(-(1-\varrho_n^2)\Lambda) = Z(1)^{\varrho_n}\exp(-(1-\varrho_n)\Lambda),$$

for the mean-variance trade-off process

$$\Lambda = \frac{1}{2}\int_0^T \lambda(Y_t)^2 dt.$$

Assuming that the taken position size q_n is of the form of $q_n = \frac{\gamma_n}{\alpha(1-\varrho_n^2)}$ for some γ_n,[17] we get by (3.2.11) and by above representation

(4.3.1)
$$\begin{aligned} p^n_{U_\alpha}(x, q_n; h) &= -\frac{1}{q_n\alpha}\log\left(\frac{u^n_{U_\alpha}(0, q_n; h)}{u^n_{U_\alpha}(0, q_n; 0)}\right) \\ &= -\frac{1}{\gamma_n}\log\left(\underbrace{\frac{\mathbb{E}^{\mathbb{P}^n}[Z^{\varrho_n}\exp(-(1-\varrho_n)\Lambda - \gamma_n h(Y_T))]}{\mathbb{E}^{\mathbb{P}^n}[Z^{\varrho_n}\exp(-(1-\varrho_n)\Lambda)]}}_{=:f(\varrho_n,\gamma_n)}\right). \end{aligned}$$

Obviously, the study of $f(\varrho_n, \gamma_n)$ will play a crucial role in the following analysis. Moreover, $f(\varrho_n, \gamma_n)$ is smooth, as we can interchange limits and integrals due to the boundedness of $\lambda(y)$ and thus of Λ, h and the existing exponential moments of Z. Hence the derivative of $f(\varrho_n, \gamma_n)$ can be computed by simply interchanging integral and differential operator. Lastly, we note that for $g(\varrho_n, \gamma_n)$ given by

(4.3.2)
$$g(\varrho, \gamma) := \frac{\mathbb{E}^{\mathbb{P}^n}[h(Y_T)Z(\varrho)\exp(-(1-\varrho^2)\Lambda - \gamma h(Y_T)]}{\mathbb{E}^{\mathbb{P}^n}[Z(\varrho)\exp(-(1-\varrho^2)\Lambda - \gamma h(Y_T)]},$$

we have that

$$g(\varrho, \gamma) = -\partial_\gamma \log(f(\varrho, \gamma)).$$

The following Lemma gives helpful properties on $g(\varrho_n, \gamma_n)$, which will be needed later. It is a consequence on Esscher transformation.

Lemma 4.3.1. ([Rob13, Lemma 7.1]) *Suppose Assumption 6 holds. For $\varrho, \gamma \in \mathbb{R}$, we have:*

i) For ϱ fixed, $g(\varrho, \gamma)$ is strictly decreasing in γ with $\lim_{\gamma\to-\infty} g(\varrho, \gamma) = \sup_{y\in E} h(y)$ and $\lim_{\gamma\to\infty} g(\varrho, \gamma) = \inf_{y\in E} h(y)$.

ii) For $p \in I'(h) := [\inf_{y\in E} h(y), \sup_{y\in E} h(y)]$ and $\varrho \in \mathbb{R}$, there exists a unique $\gamma = \gamma(\varrho)$ such that $p = g(\varrho, \gamma(\varrho))$ and the map $\varrho \mapsto \gamma(\varrho)$ belongs to $C^1(\mathbb{R})$.

Proof. We show i) for $\gamma \to \infty$:
Define a new probability measure by the following Esscher transform

$$\frac{d\tilde{\mathbb{P}}^n}{d\mathbb{P}^n} := \frac{Z^\varrho \exp(-(1-\varrho)\Lambda - \gamma h(Y_T))}{\mathbb{E}^{\mathbb{P}^n}[Z^\varrho \exp(-(1-\varrho)\Lambda - \gamma h(Y_T))]}.$$

We then get that

$$\partial_\gamma g(\varrho, \gamma) = \mathbb{E}^{\tilde{\mathbb{P}}^n}[h(Y_T)]^2 - \mathbb{E}^{\tilde{\mathbb{P}}^n}[h(Y_T)^2] = -\mathrm{Var}^{\tilde{\mathbb{P}}^n}[h(Y_T)] < 0,$$

[17] Later, we will see that this assumption on the representation of q_n is legitimated.

hence g is strictly decreasing. Setting $\underline{h} := \operatorname{essinf}_{\mathbb{P}}[h(Y_T)] = \inf_{y\in E} h(y)$ (we haven't seen this equality yet, see later in the note after Theorem 4.3.1 for details and verification) gives that $g(\varrho,\gamma) \geq \underline{h}$.
Choose $m > 0$ such that $\mathbb{P}[h(Y_T) - \underline{h} < m] > 0$ as well as $\mathbb{P}[h(Y_T) - \underline{h} \geq m] > 0$. This gives

$$\begin{aligned} g(\varrho,\gamma) &= \underline{h} + \frac{\mathbb{E}^{\mathbb{P}^n}[(h(Y_T) - \underline{h})Z^{\varrho}\exp(-(1-\varrho)\Lambda - \gamma(h(Y_T) - (\underline{h}+m)))]}{\mathbb{E}^{\mathbb{P}^n}[Z^{\varrho}\exp(-(1-\varrho)\Lambda - \gamma(h(Y_T) - (\underline{h}+m)))]} \\ &\leq \underline{h} + m + \frac{K}{\mathbb{E}^{\mathbb{P}^n}[Z^{\varrho}\exp(-(1-\varrho)\Lambda - \gamma(h(Y_T) - (\underline{h}+m)))\mathbf{1}_{h(Y_T)<\underline{h}+m}]}, \end{aligned}$$

for $\gamma > 0$ and K large enough. Fatou's Lemma implies that

$$\limsup_{\gamma\to\infty} g(\varrho,\gamma) \leq \underline{h} + m.$$

Letting m tend to 0 yields the desired result. The case for $\gamma \to -\infty$ follows in a similar way.
For ii), we note that part i) gives the existence of a unique $\gamma(\varrho)$ such that for all $p \in I'(h)$, we have: $p = g(\varrho,\gamma(\varrho))$. The result now follows from the Implicit Function Theorem by the smoothness of $g(\varrho_\gamma)$ and by $\partial_\gamma g(\varrho,\gamma) \neq 0$. □

4.3.2 Pricing in the Large Claim Limit

Recalling the definition of the large claim limit from Section 4.2.1, we finally investigate the value function in the joint limit of $q_n \to \infty$ and $\varrho_n \to 1$ and the respective pricing results. We shall establish this for $U \in \mathcal{U}_\alpha$.
It turns out in the sequel that it is convenient to express the position size q_n in terms of the correlation ϱ_n and the risk aversion parameter α and an exogenous factor γ_n. Hence following representation is (for the moment being) legitimated and not artificially forced.

$$q_n = \frac{\gamma_n}{\alpha(1-\varrho_n^2)} \text{ for some } \gamma_n, \text{ where } \begin{cases} \text{(i) } \gamma_n \to 0 \text{ but } \frac{\gamma_n}{1-\varrho_n^2} \to \infty. \\ \text{(ii) } \gamma_n \to \gamma > 0. \\ \text{(iii) } \gamma_n \to \infty. \end{cases} \tag{4.3.3}$$

Accordingly, γ_n can be seen as an exogenous parameter describing the relationship between the hedging error and the position size. Choosing q_n in this way for some $\alpha, \gamma_n, \varrho_n$, we fall directly into regime of (4.0.1) if $\gamma_n \to \gamma$, as we can identify for each fixed n

$$\underbrace{q_n}_{\text{position size}} \times \overbrace{\alpha}^{\text{risk aversion}} \times \underbrace{(1-\varrho_n^2)}_{\text{hedging error}} = \overbrace{\gamma_n}^{\text{const.}}.$$

But notice that we did not verify this relationship yet - this is exactly our goal now.
For this, we define in the limiting market a new probability measure $\mathbb{Q} \sim \mathbb{P}$ by setting

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := Z(1) = Z.$$

Note. $\mathbb{Q}$ is the unique martingale measure in the complete model, as seen in Chapter 3.

The next theorem is our main theorem and gives us explicit large claim limiting prices for the three different regimes.

Theorem 4.3.1. ([Rob13, Proposition 5.3]) *Let $\alpha > 0$, $U \in \mathcal{U}_\alpha$ and $x \in \mathbb{R}$. Suppose Assumption 6 holds. Then the average utility indifference price for the three regimes in (4.3.3) is given in the limit by*

$$\lim_{n\to\infty} p_U^n(x, q_n; h) = p_\alpha := \begin{cases} \text{(i) } \mathbb{E}^{\mathbb{Q}}[h(Y_T)] \\ \text{(ii) } -\frac{1}{\gamma}\log \mathbb{E}^{\mathbb{Q}}[\exp(-\gamma h(Y_T))] \\ \text{(iii) } \operatorname{essinf}_{\mathbb{P}}[h(Y_T)] = \inf_{y\in E} h(y). \end{cases}$$

Remark 4.3.2.

- Notice that the equality in (iii) holds due to the continuity of h and $\mathbb{P}[Y_T \in (l', u')] > 0$ for any $(l', u') \in E$. As we will see later in the proof of above theorem, the result for (iii) holds also when $\varrho_n \equiv \varrho$ and $\gamma_n \to \infty$, i.e. in a constant kept market when only position size are allowed to vary. But we have seen in Theorem 4.2.3 that in such a setting, prices converge to the minimal arbitrage-free price. Therefore Theorem 4.2.3 together with Theorem 4.3.1 imply that for each n

$$\inf_{\mathbb{Q}^n\in\mathcal{M}^n} \mathbb{E}^{\mathbb{Q}^n}[h(Y_T)] = \inf_{y\in E} h(y).$$

  and hence the interval $I(h)$ of arbitrage-free prices coincides with $I'(h)$.

- We recall that in Theorem 4.2.3 we have seen the **large position effect** for the very first time, that was

$$\lim_{n\to\infty} p_U(x, q_n; h) = \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}\left[h(Y_T)\right].$$

  Above theorem extends the large position effect to a framework, where the markets are also allowed to vary and gives even preciser limiting results depending on the exogenous factor γ_n.

- For a fixed absolute risk aversion $\alpha > 0$, and assuming $\gamma_n \to 0$, meaning that if, roughly speaking,

$$\text{position size} \times \text{hedging error} \approx 0,$$

  then we recover in the limit the arbitrage-free price under the martingale measure from the complete model, as seen in Chapter 3. Hence this regime can be treated as the small claim limit.
  Heuristically, this can be justified by noticing that as the position size increases, hedging errors become overproportional small, hence almost negligible and the investor acts in an nearly complete model where she is willing to pay the arbitrage-free Black-Scholes price. Accordingly, this overproportional behavior cancels out all incompleteness in the limiting price.

  On the other hand, if $\gamma_n \to \gamma \neq 0$, we have that

$$\text{position size} \times \text{hedging error} \approx \text{const} \approx \gamma,$$

  and a slightly modified version of the arbitrage-free price turns up. Therefore exactly in this regime, the large claim limit arises endogenously. In fact, the limiting price is given by the canonical exponential utility indifference price.
  The reasoning for this can be given as follows: On each fixed market, we have that the hedging error is not negligible and behaves (up to constants) inversely/proportional to the current position size. Hence the investor is not anymore able to remove the incompleteness by just increasing her position

size. This substantial incompleteness procreates and has a significant impact on the limiting price. Moreover, the resulting limiting price coincides with the price an investor would pay having the canonical exponential utility $U_\alpha \in \mathcal{U}_\alpha$.

Lastly, if $\gamma_n \to \infty$, then by varying the markets, the inverse relationship between 'hedging error' and 'position size' becomes adjusted and interrupted by a constant γ_n growing to infinity, meaning that

$$\frac{\text{position size}}{\gamma_n} \approx \frac{1}{\text{hedging error}}.$$

Hence, as $\gamma_n \to \infty$, the position size has to grow much faster in order to get a lower hedging error. This overproportionality gives us a somehow extreme price, the superreplication price. In other words: Hedging errors procreate and pose an overall large risk. Accordingly, investors are only willing to pay the lowest arbitrage-free price.

Let's turn our attention to the proof of this powerful theorem.

Proof. The idea is to study $f(\varrho_n, \gamma_n)$ in the three regimes. Doing this, we'll apply Taylor expansion for the first two cases and for the third case, we'll apply a convergence argument.
Recall the representation of $p^n_{U_\alpha}(x, q_n; h^n)$ and $f(\varrho_n, \gamma_n)$ in (4.3.1).
We consider case (i): Taylor expansion around $(\varrho_n, 0)$ gives

$$f(\varrho_n, \gamma_n) = 1 + \gamma_n \underbrace{\partial_\gamma f(\varrho_n, 0)}_{|.| \le \sup_{y\in E} |h(y)|} + \frac{1}{2}\gamma_n^2 \underbrace{\partial^2_{\gamma\gamma} f(\varrho_n, \xi_n)}_{|.| \le \sup_{y\in E} |h(y)|^2} \quad \text{for some } 0 \le \xi_n \le \gamma_n,$$

as $\partial_\gamma \log(f(\varrho,\gamma)) = \frac{\partial_\gamma f(\varrho,\gamma)}{f(\varrho,\gamma)}$ and $f(\varrho,\gamma) < 1$ implying $|\partial_\gamma f(\varrho,\gamma)| \le |g(\varrho,\gamma)| \le \sup_{y\in E}|h(y)|$. Similar calculations give $|\partial^2_{\gamma\gamma} f(\varrho_n, \xi_n)| \le \sup_{y\in E} |h(y)|^2$. This implies

$$\begin{aligned}
\lim_{n\to\infty} p^n_{U_\alpha}(x, q_n; h) &= \lim_{n\to\infty} -\frac{1}{\gamma_n} \log(f(\varrho_n, \gamma_n)) \\
&= \lim_{n\to\infty} -\frac{1}{\gamma_n} \left(\gamma_n \partial_\gamma f(\varrho_n, 0) + \frac{1}{2}\gamma_n^2 \partial^2_{\gamma\gamma} f(\varrho_n, \xi_n) \right) \\
&= -\partial_\gamma f(1, 0) = \mathbb{E}^{\mathbb{P}}[Z h(Y_T)],
\end{aligned}$$

where we used $\log(1+x) \approx x$ for small x and the continuity of $\partial_\gamma f(\varrho_n, \gamma_n)$.
The case (ii) is handeled analogously with Taylor expansion around (ϱ_n, γ).

$$f(\varrho_n, \gamma_n) = f(\varrho_n, \gamma) + (\gamma_n - \gamma)\partial_\gamma f(\varrho_n, \gamma) + \frac{1}{2}(\gamma_n - \gamma)^2 \partial^2_{\gamma\gamma} f(\varrho_n, \xi_n) \text{ for some } 0 \le \xi_n \le \gamma_n,$$

giving

$$\lim_{n\to\infty} p^n_{U_\alpha}(x, q_n; h) = -\frac{1}{\gamma} \log\left(f(1,\gamma)\right) = -\frac{1}{\gamma} \log\left(\mathbb{E}^{\mathbb{Q}}[\exp(-\gamma h(Y_T))]\right).$$

Finally for case (iii), we set $\underline{h} := \operatorname{essinf}_{\mathbb{P}}[h(Y_T)]$. Then clearly $\liminf_{n\to\infty} p^n_{U_\alpha}(x, q_n; h) \ge \underline{h}$. Let $m > \underline{h}$ such that $\mathbb{P}[h(Y_T) < m] > 0$, then

$$\mathbb{E}^{\mathbb{P}}\left[Z^{\varrho_n} \exp\left((1-\varrho_n)\Lambda - \gamma_n h(Y_T)\right)\right] \ge e^{-\gamma_n m} \mathbb{E}^{\mathbb{P}}\left[Z^{\varrho_n} \exp\left(-(1-\varrho_n)\Lambda\right) \mathbf{1}_{h(Y_T)<m}\right].$$

This implies that

$$\limsup_{n\to\infty} p^n_{U_\alpha}(x, q_n; h) \le m.$$

The result finally follows by letting $m \to \underline{h}$. □

4.3.3 Determining Optimal Quantities

In this section, we want to verify the heuristic equation (4.0.1), meaning that we want to study whether our assumption on the form of $q_n = \frac{\gamma_n}{\alpha(1-\varrho_n^2)}$ was reasonable.
We have seen that the interval $I(h)$ of arbitrage-free prices for $h(Y_T)$ is given by

$$I(h) = \left[\inf_{\mathbb{Q}^n \in \mathcal{M}^n} \mathbb{E}^{\mathbb{Q}^n}[h(Y_T)], \sup_{\mathbb{Q}^n \in \mathcal{M}^n} \mathbb{E}^{\mathbb{Q}^n}[h(Y_T)] \right] = \left[\inf_{y \in E} h(y), \sup_{y \in E} h(y) \right]$$

Fix $n \in \mathbb{N}$, let $p_n \in I(h)$ and assume that an investor can buy an arbitrary number of claims for this fixed price p_n, meaning that there is enough liquidity in the market.

Note. Notice that at this point, it is not yet clear, from where this liquidity comes from.

Question. A natural problem that arises is to *determine* the utility based **optimal quantity** [18]:

$$q_n \in \operatorname{argmax}_{q \in \mathbb{R}} u^n_{U_\alpha}(x - qp_n, q; h). \tag{4.3.4}$$

The following theorem gives us an answer to this problem and even justifies the relation from (4.0.1).

Theorem 4.3.2. ([Rob13, Proposition 5.5]) *Let Assumption 6 hold and let $p_n \in I(h)$. Then the unique q_n solving (4.3.4) satisfies*

$$\alpha q_n (1 - \varrho_n^2) = \gamma_n,$$

where γ_n is uniquely determined by $p_n = g(\varrho_n, \gamma_n)$.
If $\varrho_n \to 1$, then for any subsequence $\{n_k\}_{k \in \mathbb{N}}$, we have

$$\lim_{k \to \infty} |q_{n_k}| = \infty \iff \lim_{k \to \infty} \frac{|p_{n_k} - \hat{p}|}{1 - \varrho_{n_k}^2} = \infty, \tag{4.3.5}$$

where $\hat{p} := \mathbb{E}^{\mathbb{P}}[Z(1)h(Y_T)] = \mathbb{E}^{\mathbb{Q}}[h(Y_T)]$ is the unique arbitrage-free price in the complete market.
Moreover, if we have the convergence of $p_n \to p$ for some arbitrage-free price $p \in I(h)$, then

$$\lim_{n \to \infty} \alpha q_n (1 - \varrho_n^2) = \gamma,$$

where γ uniquely solves $p = g(1, \gamma)$.
Lastly,

$$\gamma \neq 0 \iff p \neq \hat{p}.$$

Remark 4.3.3.

- Theorem 4.3.2 essentially states that when an investor purchases optimal quantities (in the sense of (4.3.4)), the third regime given by case (iii) in (4.3.3) will never appear. Case (i) arises if $p_n \to \hat{p}$, where $\hat{p} = g(1, 0)$ is the unique arbitrage-free price in the complete market. Else, case (ii) arises and $p_n \to p \in I(h)$, where $p = g(1, \gamma)$ for $\gamma \neq 0$.

[18] For exponential utility functions, existence of a unique maximizer q_n is proven in a general framework. For further details, see [IJS05, Theorem 3.1].

- Furthermore (4.3.5) gives us a necessary and sufficient condition for the appearance of the large claim limit, given hedging errors become negligible. It states that $q_n \to \infty$ if and only if the deviance of the average utility indifference price with respect to the Black-Scholes price rescaled by the hedging error explodes. In regime (i), we have that $p_n \to \hat{p}$, hence large claims do not appear, unless the markets become overproportionally complete and then investors take profit from the arbitrage opportunity. Obviously, in regime (ii) and (iii), large claims arise.

- Moreover, Theorem 4.3.2 gives a reasoning of the heuristic relationship derived in (4.0.1): for a price $p_n \in I(h)$, we have that the utility based optimal quantity q_n has the form of

$$q_n = \frac{\gamma_n}{\alpha(1-\varrho_n^2)},$$

  where γ_n is uniquely given by $p_n = g(\varrho_n, \gamma_n)$. However, we have never addressed the question of finding a counterpart for becoming optimally positioned, especially for prices $p_n \neq \hat{p}$.

Proof. [IJS05, Theorem 3.1] and also heuristic arguments gives that the optimal q_n must satisfy the following first order condition

$$g(\varrho_n, \gamma_n) = p_n \overset{!}{=} -\partial_q \log\left(\left(\frac{u^n_{U_\alpha}(0, q_n; h)}{u^n_{U_\alpha}(0, q_n; 0)}\right)^{1-\varrho^2}\right) = g(\varrho_n, \alpha q_n(1-\varrho_n^2)),$$

where we use the definition of $g(\varrho, \gamma)$ for the third equality and Lemma 4.3.1 in the first equality for the existence of a unique γ_n, hence

$$\gamma_n = \alpha q_n(1-\varrho_n^2).$$

Let $\varrho_n \to 1$ and assume that $\sup_n \frac{|p_n - \hat{p}|}{1-\varrho_n^2} < \infty$. Then again by Lemma 4.3.1, we get that $\gamma_n \to 0$ and hence by Taylor expansion

$$\frac{|p_n - \hat{p}|}{1-\varrho_n^2} = \frac{g(\varrho_n, \gamma_n) - g(1,0)}{1-\varrho_n^2} \approx -\frac{1}{1+\varrho_n}\partial_\varrho g(1,0) + \frac{\gamma_n}{1-\varrho_n^2}\partial_\gamma g(1,0).$$

From this we get the claimed equivalence. Finally, assume that $p_n \to p \neq \hat{p}$. Then by continuity, we have that $\gamma_n \to \gamma$ where γ satisfies $p = g(1,\gamma)$. Finally we have that $\gamma \neq 0 \iff p \neq \hat{p}$ by Lemma 4.3.1. ☐

4.3.3.1 Finding a Counterparty for Becoming Optimal Positioned

Question. *How* is it possible to find a buyer/seller at a price $p_n \approx p_\alpha \neq \hat{p}$, i.e. at a price which is not equal to the arbitrage-free price?

A way to try to explain this phenomenon is to introduce so-called partial-equilibrium price quantities. A very good reference on which also the following investigations are based is given by [AŽ10].

Definition 4.3.1. A pair (q_n, p_n), where $p_n \in I(h)$ and $q_n \in \mathbb{R} \setminus \{0\}$ is called a **partial-equilibrium price quantity** (short: **PEPQ**) in the n^{th} market if

1. $q_n \in \operatorname{argmax}_{q\in\mathbb{R}} \left(e^{\alpha q_n p_n} u^n_\alpha(q_n; h, X)\right)$ and
2. $-q_n \in \operatorname{argmax}_{q\in\mathbb{R}} \left(e^{\delta q_n p_n} u^n_\delta(q_n; h, X')\right)$,

where

$$u^n_\alpha(q_n; h; X) := u^n_{U_\alpha}\left(0, q_n; h + \frac{X}{q_n}\right)$$

is the value function for holding q_n units of $h(Y_T)$ and one unit of the bounded T-claim X. X' is another bounded T-claim.
A price $p_n \in I(h)$ is called **partial-equilibrium price** in the n^{th} market (short: **PEP**) if there exists q_n such that (q_n, p_n) is a PEPQ.

In other words: (q_n, p_n) is a PEPQ if, for the price p_n, it is optimal for the δ-risk averse investor to sell q_n units of $h(Y_T)$ and for the α-risk averse investor to buy q_n units of $h(Y_T)$. A general result is that if $\alpha X - \delta X'$ is not replicable, then there exists a unique PEPQ (q_n, p_n), otherwise there is no PEPQ ([AŽ10, Theorem 5.8]).

The name has its origin as, in a two-agents economy, the agents can not only agree on the price but also upon the quantity of their transaction meaning that given a certain endogenous and arbitrage-free price $p_n \in I(h)$, the agents enter into a partial-equilibrium (and thus the trade will occur) if they can agree on the volume of their trade (i.e. if they find a quantity q_n such that the transaction is optimal for both parties). The equilibrium is only partial as the two agents are temporarily in an equilibrium and the whole market does not have to be in balance. [AŽ10, Section 5]

Returning to the verification of (4.0.1): Assume that $X \equiv 0$ and that we have a two-agents economy with, say, a seller (δ risk averse) and a buyer (α risk averse) [19]. Moreover assume that the seller holds a position of q_n units of $h(Y_T)$. Then this can be expressed in terms of (4.3.3) meaning that we find γ_n such that

$$X' = q_n h(Y_T) = \frac{\gamma_n}{\delta(1-\varrho_n^2)} h(Y_T) \text{ for some } \gamma_n > 0. \tag{4.3.6}$$

As $h(Y_T)$ is not replicable due to incompleteness, it follows that there exists a PEPQ $(\hat{q}_n, p_n)$ for the price p_n which satisfies the optimality conditions for $\hat{q}_n := \frac{\gamma_n'}{(1-\varrho_n^2)}$:

$$p_n = g(\varrho_n, \alpha\gamma_n') = g(\varrho_n, \gamma_n - \delta\gamma_n').$$

Indeed, the first equality comes from the fact that the unique $\hat{q}_n$ solving the optimization problem is given by

$$p_n = g(\varrho_n, \gamma_n''),$$

for $\hat{q}_n = \frac{\gamma_n''}{\alpha(1-\varrho_n^2)}$.
The second equality is due to the same reason, with a small adjustment, namely that the seller with risk aversion δ chooses the quantity she wants to sell for the price $p_n \neq \hat{p}$ such that the remaining quantity held, which is

$$\tilde{q}_n = \frac{\gamma_n}{\delta(1-\varrho_n^2)} - \hat{q}_n = \frac{\gamma_n - \delta\gamma_n'}{\delta(1-\varrho_n^2)},$$

becomes optimal in the sense, that above $\tilde{q}_n$ satisfies (4.3.4) which gives

$$p_n = g(\varrho_n, \gamma_n - \delta\gamma_n').$$

We have therefore that

$$\gamma_n' = \frac{\gamma_n}{\alpha + \delta},$$

[19] We call the two agents a priori 'seller' and 'buyer'. To be more precise, the situation is as follows: One of the agents possesses a number of claims h and tries to sell them. Given that there is an endogenous price p_n determined by the market environment, the two agents close their deal if and only if they can agree on the quantity q_n such that (p_n, q_n) is a PEPQ.

which gives us in conclusion that the two agents have agreed on the quantity

$$\hat{q}_n = \frac{\gamma_n'}{(1-\varrho_n^2)} = \frac{\gamma_n}{(1-\varrho_n^2)(\alpha+\delta)}$$

of the claim h for the price p_n, which is given by the representation

$$p_n = g\left(\varrho_n, \frac{\alpha\gamma_n}{(\alpha+\delta)}\right).$$

As $\varrho_n \to 1$, we have that $p_n \to p \in I(h)$, where

$$p = g\left(1, \frac{\alpha\gamma}{(\alpha+\delta)}\right),$$

for $\gamma_n \to \gamma$. Depending on the value of γ, we have that $p = \hat{p}$ if $\gamma = 0$ and $p \neq \hat{p}$ if $\gamma > 0$.
Clearly, the buyer enters into the regime of (4.0.1), as

$$\hat{q}_n \times (1-\varrho_n^2) \times \alpha = \frac{\gamma_n \alpha}{(\alpha+\delta)} \approx \text{ const for the n-th market.}$$

Summing up: given that the seller was already in regime of (4.0.1), which we assumed in (4.3.6), acting optimally, the buyer directly enters into the regime of (4.0.1) as well.

The message is that as long as there exists a single investor in the regime of (4.0.1), independent of the optimality of entering into this regime, it is possible for other investors to enter into the regime of (4.0.1) in an optimal way.
Given the fact that we have huge actual notional sizes, we may assume that there is always an investor in the regime of (4.0.1) and each other agent interacting or trading respectively with this investor enters into the regime of (4.0.1).

4.3.4 Monetary Errors

As we have seen in Theorem 4.2.2, the monetary error induced by the difference of utility indifference prices for utilities from the same exponential class remains bounded in the limit. Additionally, one might be interested if this is still true when we directly use the limiting price. Generally, this is not the case. Indeed, we have the following theorem:

Theorem 4.3.3. ([Rob13, Proposition 5.7]) *Let $\alpha > 0$ and let Assumption 6 hold. Then for q_n given by (4.3.3) and p_α from Theorem 4.3.1, we have that, as $\varrho_n \to 1$*

$$\limsup_{n\to\infty} q_n |p^n_{U_\alpha}(x, q_n; h) - p_\alpha| < \infty \iff \begin{cases} \text{(i) } \limsup_{n\to\infty} \frac{\gamma_n^2}{(1-\varrho_n^2)} < \infty \\ \text{(ii) } \limsup_{n\to\infty} \frac{|\gamma-\gamma_n|}{(1-\varrho_n^2)} < \infty. \end{cases}$$

Note that for simplification, we omit case (iii).
Moreover, if γ_n is chosen optimally as in Theorem 4.3.2 for a fixed $p \in I(h)$ meaning that γ_n satisfies $p = g(\varrho_n, \gamma_n)$, then monetary errors are always bounded.

Proof. We introduce the monetary error in the n^{th} market

$$\mathrm{ME}_n := q_n |p^n_{U_\alpha(x,q_n;h)} - p_\alpha| = \frac{\gamma_n}{\alpha(1-\varrho_n^2)} \left| -\frac{1}{\gamma_n} \log(f(\varrho_n, \gamma_n)) - p_\alpha \right|,$$

for $f(\varrho_n, \gamma_n)$ defined in (4.3.1) and $q_n = \frac{\gamma_n}{\alpha(1-\varrho_n^2)}$. To estimate ME_n, it is therefore necessary to approximate $\log(f(\varrho_n, \gamma_n))$ using Taylor expansion around $\log(f(1,\gamma))$.
As $\frac{1-\varrho_n}{1-\varrho_n^2} \leq 1$, we have that the terms derived by partial derivatives with respect to ϱ remain bounded. Hence the only terms contributing to the finiteness of ME_n are $\log(f(1,\gamma)), \partial_\gamma \log(f(1,\gamma)), \partial^2_{\gamma\gamma} \log(f(1,\gamma))$. Therefore we may consider the following approximation

$$\log(f(\varrho_n, \gamma_n)) \approx \log(f(1,\gamma)) + (\gamma_n - \gamma)\frac{\partial_\gamma f}{f}(1,\gamma) + \frac{1}{2}(\gamma_n-\gamma)^2 \left(\underbrace{\frac{\partial^2_{\gamma\gamma} f}{f} - \left(\frac{\partial_\gamma f}{f}\right)^2}_{-\partial_\gamma g} \right)(1,\gamma),$$

where $g(\varrho_n, \gamma_n)$ was defined in (4.3.2).
For case (i), we have $p_\alpha = -\partial_\gamma \log(f(1,0)) = \mathbb{E}^{\mathbb{Q}}[h(Y_T)]$ and $\gamma = 0$, hence above approximation yields

$$\mathrm{ME}_n = \frac{\gamma_n}{\alpha(1-\varrho_n^2)} \left| \frac{1}{2}\gamma_n \left(\partial^2_{\gamma\gamma} f - (\partial_\gamma f)^2\right)(1,0) \right|,$$

which gives the desired result. For case (ii), we have that $p_\alpha = -\frac{1}{\gamma}\log(f(1,\gamma))$ and $\gamma \neq 0$ which gives us

$$\mathrm{ME}_n = \frac{|\gamma_n - \gamma|}{\alpha(1-\varrho_n^2)} \left| \frac{\partial_\gamma f}{f}(1,\gamma) + \frac{1}{2}(\gamma_n - \gamma)\left(\frac{\partial^2_{\gamma\gamma} f}{f} - \left(\frac{\partial_\gamma f}{f}\right)^2\right)(1,\gamma) \right|,$$

which indicates the equivalence for case (ii).
For the last statement, let $p \in I(h)$ and assume γ_n is chosen optimally as in Theorem 4.3.2. If $p = \hat{p}$, then by Theorem 4.3.2 we have that $\sup_n |q_n| < \infty$, hence the assertion follows. Else, if $p \neq \hat{p}$, then again by Theorem 4.3.2, we have that $\gamma_n \to \gamma \neq 0$ where $p = g(1,\gamma)$. Hence we are in regime (ii) and the monetary error is finite if and only if $\sup_n \frac{|\gamma_n - \gamma|}{(1-\varrho_n^2)} < \infty$. Hence for $\varepsilon > 0$

$$\sup_n \frac{|\gamma_n - \gamma|}{(1-\varrho_n^2)} = \frac{|\gamma(\varrho_n) - \gamma(1)|}{(1-\varrho_n^2)} = \frac{1}{1+\varrho_n}\frac{1}{1-\varrho_n}\left| \int_{\varrho_n}^1 \gamma'(\tau)d\tau \right| \leq \frac{|\gamma'(1)|+\varepsilon}{2} < \infty,$$

where we used the fact that the map $\varrho \mapsto \gamma(\varrho)$ belongs to $C^1(\mathbb{R})$. □

4.3.5 Algorithm for Finding the Optimal Quantity $q^\star$

Above discussion gives us an algorithm for finding the optimal quantity $q^\star$.
Assume, that the claim $h(Y_T)$ can be purchased for a price p where we assume that $p \in I(h)$ and further assume a fixed and known ϱ (in practice, this can be estimated by the use of statistical methods). Note, that this does not imply that we know the regime in which the agent is acting (under the assumption of optimal acting, both regime (i) and regime (ii) are possible), as we do not have to know this for determining optimal quantities.
Then Lemma 4.3.1 (ii) gives us the existence of a $\gamma(\varrho)$ such that $p = g(\varrho, \gamma(\varrho))$. Together with Theorem 4.3.2, we have the optimality property of $\gamma(\varrho)$ and moreover we know that $\gamma(\varrho)$ satisfies $\gamma(\varrho) = \alpha q^\star(1-\varrho^2)$.

Hence we just have to solve

$$p = g(\varrho, \gamma(\varrho))$$

for $\gamma(\varrho)$, i.e.

$$p = \frac{\mathbb{E}^{\mathbb{P}}[h(Y_T) Z^{\varrho} \exp(-(1-\varrho)\Lambda - \gamma(\varrho) h(Y_T))]}{\mathbb{E}^{\mathbb{P}}[Z^{\varrho} \exp(-(1-\varrho)\Lambda - \gamma(\varrho) h(Y_T))]}.$$

Define a new probability measure $\mathbb{Q}^{\varrho}$ by

$$\frac{d\mathbb{Q}^{\varrho}}{d\mathbb{P}} = Z(\varrho) = \mathcal{E}\left(-\varrho \int_0^{\cdot} \lambda(Y_t) dW_t\right)_T.$$

Then above condition on $\gamma(\varrho)$ reduces to

$$p = \frac{\mathbb{E}^{\mathbb{Q}^{\varrho}}[h(Y_T) \exp(-(1-\varrho^2)\Lambda - \gamma(\varrho) h(Y_T))]}{\mathbb{E}^{\mathbb{Q}^{\varrho}}[\exp(-(1-\varrho^2)\Lambda - \gamma(\varrho) h(Y_T))]}. \tag{4.3.7}$$

Hence, finding a solution in (4.3.7) requires to follow the following algorithm: Start with two initial values for γ, γ_{low} and γ_{high}, say. As the right-hand side in (4.3.7) is monotonic in γ, we can apply a nesting procedure.

Algorithm 1 Algorithm for determining $q^\star$ and $p_U(x, q^\star; h)$

Given data: $p \in I(h)$, γ_{low}, γ_{high}, Tol

$\gamma_{\text{temp,low}} \leftarrow \gamma_{\text{low}}$
$\gamma_{\text{temp,high}} \leftarrow \gamma_{\text{high}}$
Error $\leftarrow$ 2Tol
while Error $>$ Tol **do**
  $\gamma_{\text{temp}} \leftarrow \frac{1}{2}(\gamma_{\text{temp,high}} + \gamma_{\text{temp,low}})$
  $p_{\text{temp}} \leftarrow g(\varrho, \gamma_{\text{temp}})$
  if $p_{\text{temp}} > p$ **then**
    $\gamma_{\text{temp,low}} \leftarrow \gamma_{\text{temp}}$
  else
    $\gamma_{\text{temp,high}} \leftarrow \gamma_{\text{temp}}$
  end if
  Error $\leftarrow |p_{\text{temp}} - p|$
end while
$q^\star \leftarrow \frac{\gamma_{\text{temp}}}{\alpha(1-\varrho^2)}$
$p_U(x, q^\star; h) \leftarrow -\frac{1}{\gamma_{\text{temp}}} \log(f(\varrho, \gamma_{\text{temp}}))$

Algorithm 1 gives us the bifurcation procedure of finding the optimal position size $q^\star$ given a given market price of $p \in I(h)$. Moreover, we can then calculate the utility indifference price $p_U(x, q^\star; h)$ for which the agent is indifferent between holding $q^\star$ claims of $h(Y_T)$ or not. Depending on the order of p and $p_U(x, q^\star; h)$, i.e. $p < p_U(x, q^\star; h)$ resp. $p > p_U(x, q^\star; h)$ the agent will take in a long resp. short position.

4.4 Example

We end up this chapter by giving a concrete example. Part of our investigation is based on the `Mathematica` code published by S. Robertson on his homepage, see [Rob]. In what follows, we assume that the agent's utility function is given by the canonical exponential utility function $U_\alpha = -\frac{1}{\alpha}e^{-\alpha x}$ for some absolute risk aversion $\alpha > 0$ and that we are in the setting the basis risk model, hence S_t^n and Y_t are given by two (correlated) geometric Brownian motions.
A big drawback of this approach is that we are not able to price standard Call options. Hence we turn our attention directly to Put options.

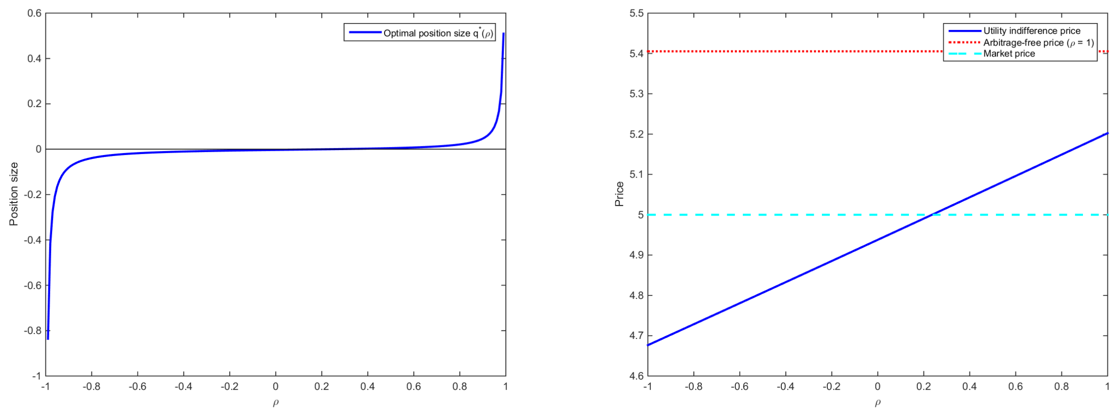
4.4.1 Put Option

4.4.1.1 Finding the Optimal Quantity

We let $h(Y_T) = (K - Y_T)^+$. Of course, Assumption 6 is satisfied, as Y_t is driven by a geometric Brownian motion and thus non-negative.

Parameters - We choose the following parameters:
$K = 20,\ T = 1,\ t = 0\ \nu = 0.05,\ \mu = 0.06,\ \eta = 0.3,\ \sigma = 0.4,\ Y_0 = 15,\ \alpha = 3,\ p = 5$

With this parameters and following Algorithm 1, we produced the following plots. Note that in a first step, we solved $p = g(\varrho, \gamma(\varrho))$ for $\gamma(\varrho)$ and by this we can then calculate the optimal position size $q^\star$ and the average utility indifference price $p_{U_\alpha}(x, q_n; h)$.



(a) Change of optimal position size when $\varrho \to \pm 1$. (b) Change of the price when $\varrho \to \pm 1$.

Figure 4.4.1: Price and position dynamics when $\varrho \to \pm 1$ for a Put option with strike $K = 20$, $Y_0 = 15$.

Figure 4.4.1 shows that under these specific parameters and with an a priori fixed market price of $p = 5$ per unit of $h(Y_T)$, then whenever the optimal position size is positive, we have that the respective average utility indifference price is larger than the market price. On the other hand side, if the optimal quantity is negative, we have that $p_{U_\alpha}(x, q; h) \leq 5$.
This is nothing else than what we should have expected: As long as a specific agent is in a market (by this we specify ϱ) where it is optimal to take a long position in the claim, it has to hold for sure that the average utility indifference price is larger than the price for which she is taking a long position as the

long position indicates an increase of her expected utility at time T. In contrast to that, if it is optimal to shorten the claim $h(Y_T)$, her utility indifference price has to be smaller than this price.
We want to point out that as $\varrho \to 1$, we see that $p_{U_\alpha}(x,q;h) \to p \approx 5.2 \neq \hat{p} \approx 5.4053$, where $\hat{p}$ is the arbitrage-free Black-Scholes price. This means that we are either in regime (ii) or (iii) (as if we were in regime (i), we would see the arbitrage-free price). But we have seen in Remark 4.3.3 that regime (iii) won't appear if we purchase optimal quantities. Hence we are for sure in regime (ii). By this (referring to Remark 4.3.2), we find that hedging errors are not negligible and are (up to a constant) inverse to the current position size. Moreover, the position size increases to $\pm\infty$ as $\varrho \to \pm 1$. This is of course due to the fact that $p \neq \hat{p}$ as we are in regime (ii) and the agent profits from the arbitrage opportunity. Here we see also, that the large claim limit arises endogenously when letting the markets converge to a limiting market and the resulting price is not equal to the arbitrage-free Black-Scholes price.
We have also given reasons how one could engage a seller/buyer for a price $p_{U_\alpha} \approx p \neq \hat{p}$ using the notion of partial equilibrium price quantities.
It should be clear that all these investigations highly depend on the market price $p = 5$ as the agent aligns her position size according to the market price.

4.4.1.2 Pricing

In the sequel, we want to investigate the pricing behavior of the Put option in the given framework. For completeness, we record again the chosen parameter:

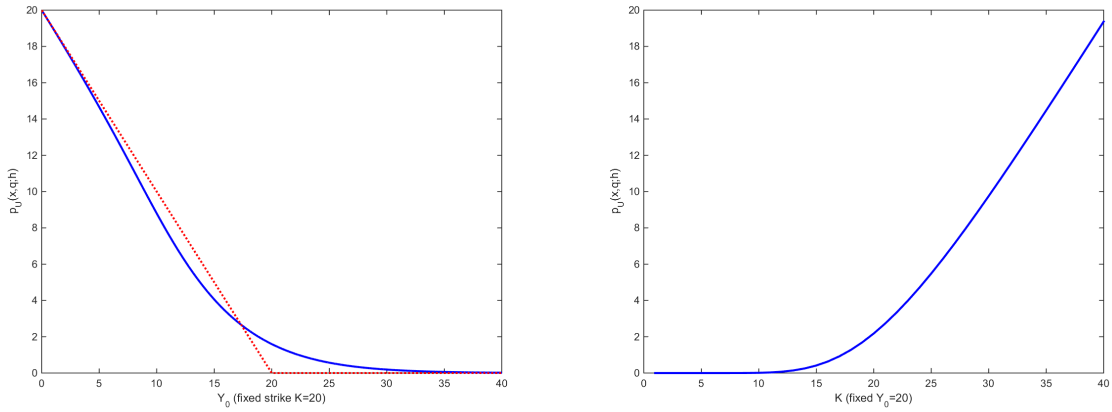
Parameters - We choose the following parameters:
$T = 1,\ t = 0\ \nu = 0.05,\ \mu = 0.06,\ \eta = 0.3,\ \sigma = 0.4,\ ,\alpha = 3, \gamma = 0.2$ (i.e. $q \approx 0.1851$), $\varrho = 0.8$

Figure 4.4.2 shows on the left the shape of the price dynamics of a Put option (long position) with a fixed strike of $K = 20$ where we included (dotted line) the payoff profile of the Put option at time T. By naked eyes, we see that the average utility indifference prices differ significantly from the Black-Scholes prices. The reason originates from the large position effect, i.e. the power of a large investor (here even $q = 0.1851$ is enough (!)) being able to push prices down. On the right-hand side we provide the dynamics of the price with respect to the strike K which is what we should have expected and already seen in Chapter 3.

4.4.1.3 Pricing Error and Large Position Effect

We turn our attention to the pricing deviation: Figure 4.4.3 shows the pricing error resulted from pricing with the arbitrage-free Black-Scholes price instead of pricing with the true parameters ϱ and q resp. γ and the respective average utility indifference price $p_U(x,q;h)$ in the actually incomplete market.
We want to direct the attention to the fact that as $\gamma \to 0$, we recover the small claim limit, prices converge to the Black-Scholes price and the resulting error vanishes. But already for $\gamma = 0.2$, the deviation from Black-Scholes price is significant as seen in the last section.

In contrast to that, we plotted in Figure 4.4.4 the pricing error with respect to the limiting price from regime (ii) (we have set $\gamma = 20$). Of course as $\gamma \to 20$, the pricing deviation vanishes.



(a) Price dynamics with respect to Y_0.

(b) Price dynamics with respect to K.

Figure 4.4.2: Different price dynamics for Put option.

But we want to point out that the deviation is way smaller in regime (ii) than in regime (i). The reason for this lies in the fact that the limiting price p_α from regime (ii) takes into account the large position effect, while Black-Scholes does not.
In summary, it can therefore be concluded that as soon as we are dealing with positive position sizes, the resulting pricing error when pricing with the limiting price from regime (ii) instead of the actual average utility indifference price is very small (even almost negligible).
This is a great insight - investors should always take $-\frac{1}{\gamma}\log(f(\varrho,\gamma))$ as a reference price.

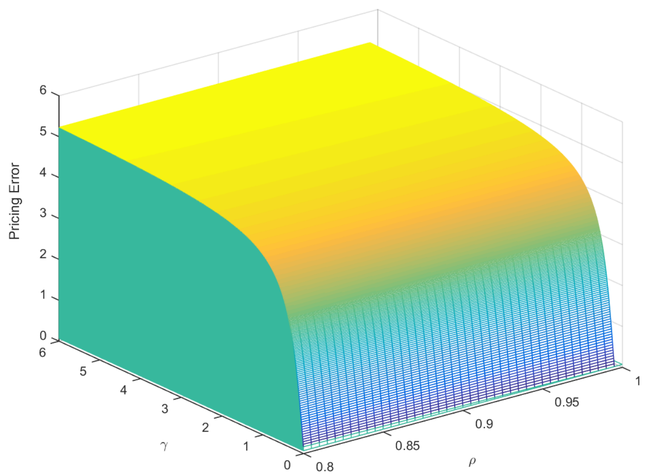


Figure 4.4.3: Plot of pricing error for a Put option in regime (i).

Lastly, we provide in Figure 4.4.5 an illustration of how fast the large position effect comes into play. Also here we see that for our initial choice of $\gamma = 0.2$, the deviation is already very significant.

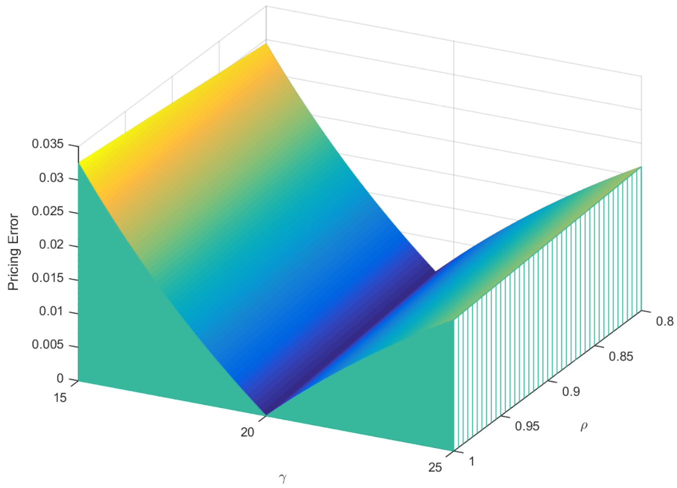


Figure 4.4.4: Plot of pricing error for a Put option in regime (ii).

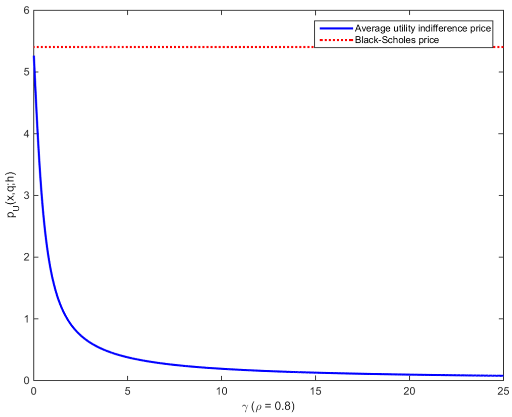


Figure 4.4.5: Large position effect for a Put option.

4.5 Conclusion

Based on [Rob13] by Scott Robertson, the utility indifference pricing problem is studied in the large claim limit, meaning that working on a sequence of markets, the position size, the claim as well as the markets are allowed to vary under some suitable assumptions. It is crucial to allow the markets to vary to get a deeper understanding of the effect of large positions on average utility indifference prices.

We have presented in a first step some convergence results for exponential and power law utilities supporting wealths on the real line. Surprisingly, the individual utility function has no impact on the limiting price but rather the rate of decay for large negative wealths has an impact on the price. We have shown that price differences will vanish in the limit for utilities belonging to the same (exponential) class. Hence, one could directly work with the canonical example for pricing purposes. The same holds true for utilities with a power-like decay for large negative wealths with a slight adjustment in the speed at which the position size grows to infinity.

In the case of a constant-kept market where only the position size is allowed to vary, average utility indifference prices converge to the minimal arbitrage-free price, i.e. the superreplication price. This is of course unsatisfactory and that is the reason why one allows also the markets to vary.
We have also seen that the price per unit for q units of the claim which an investor having exponential utility with parameter α would pay coincides with the price an agent having exponential utility with parameter $q\alpha$ would pay for one unit of the claim. By this we than also found that the limit as risk aversion goes to infinity of the average utility indifference price is also given by the lowest arbitrage-free price.
Additionally, we have studied extensively a stochastic factor model, from which we could apply the results to the basis risk model. We then distinguished between different drivers resp. regimes for the position size q_n to grow. Given the distinction of the drivers, limiting prices are established and give surprisingly different results, which are not necessarily equal to the arbitrage-free price from the complete model, although the sequence of markets is assumed to converge to a market with vanishing hedging error, hence asymptotically complete. This was a generalization to the afore-mentioned limiting price under in constant kept market. Roughly speaking, depending on the speed of becoming complete with respect to the rate of position growth, the limiting price becomes equal to either (i) the arbitrage-free Black-Scholes price, or (ii) the lowest arbitrage-free price or (iii) a price which is somehow in between of the previous extreme prices. It turns out that this price coincides with the price an exponential utility investor would pay.
As a consequence, we then have concluded that the large claim limit arises endogenously in regime (ii), while in regime (i), the investor would not hold a large claim (it can even be compared with the small claim limit) and in regime (iii) she will hold a large claim but only for the lowest possible price.
Of course, it is crucial to understand how one can find a seller/buyer for a price which is not equal to the arbitrage-free price from the complete model. This was the last question which we addressed and which is handled by the use of partial equilibrium price quantities - a tool from economics which gives a way of studying partial equilibria in simultaneously prices and quantities and does not pay attention to other factors/markets.
In the analysis, one finds that, given an investor is in the regime of (4.0.1), each additional agent interacting optimally with this specific investor enters directly into this regime. The reasoning for this was given by the fact that, nowadays, we have huge notional sizes. Of course this assertion is quite bold.
We ended the Chapter with numerical investigation of a concrete example. We gained great insight in the deviation of the respective limiting price. Namely, as long as we are dealing with positive positions, the deviation to the limiting price of regime (ii) is way smaller than the error to Black-Scholes prices.
A drawback of the approach is that only (uniformly) bounded claims can be priced in this framework.

Chapter 5

Comparison of the Two Approaches, Conclusion

In this chapter, we shall compare the two approaches given in Chapter 3 and Chapter 4. Let us start by a short overview on the approaches.

5.1 Approaches

The main difference between Chapter 3 and Chapter 4 is, as the name states, that they tackle the problem of finding the value function from two very different point of views. Recalling the initial goal of examining the value function $u_U(x, q; h)$ in the basis risk model, we studied in Chapter 3 the value function $u_U(x, q; h)$ (and by this the average utility indifference price $p_U(x, q; h)$) in the limit when the position size tends to zero, i.e. in the small claim limit. Surprisingly, we found than the optimal strategy differs barely from the optimal strategy from the complete model - the only difference lies in the magnitude of a Delta hedge term. By this, also prices are similar to the Black-Scholes prices. We have seen an expansion of the average utility indifference price up to order q whose first term coincides with the Black-Scholes term. Our numerical investigations showed that the second order term is negligible for small position sizes, at least in our treated examples. We proved and examined the small claim limit under power law utility and also presented results in the case of exponential utility.

In contrast to that, we studied in Chapter 4 the value function in the large claim limit by considering a sequence of markets $(\Omega^n, (\mathcal{F}_t^n)_{0 \leq t \leq T}, \mathbb{F}^n, \mathbb{P}^n)$ where in each market we have a basis risk model. The correlation ϱ_n and the position size q_n are allowed to vary with the constraints that $q_n \to \infty$ and $\varrho_n \to 1$. We have seen reasons for allowing the markets to vary.

We then have shown that as the number of claims is growing to infinity, the individual utility function plays no role for the resulting price but more the rate of decay for large negative wealths and the limiting price coincides with the limiting price of an investor with an exponential utility function with this specific rate of decay.

In one of our main results, we found explicit formulae for the average utility indifference price in the limit in three different regimes which differ by the speed of q_n growing to infinity. Furthermore, we have also provided cases, where the large claim limit arises endogenously.

5.2 Results

We have established several examples to highlight our investigations in Chapter 3 and Chapter 4. These examples have not produced any big surprises - we could describe the outcome using the afore-developed theory. But we never addressed the question of consistency between the two approaches which we will be our focus in the sequel.

First, note that a comparison of the explicit pricing formula presented in Chapter 3 for exponential utility functions (Theorem 3.2.2) with the one studied in Chapter 4 is worthless as they are completely equivalent when identifying γ with $\alpha q(1-\varrho^2)$ in

$$p_U(x,q;h) = -\frac{1}{\alpha q(1-\varrho^2)} \log\left(\mathbb{E}^{\mathbb{Q}}\left[\exp(-\alpha q(1-\varrho^2)h(Y_T))\right]\right).$$

However, we also provided an expansion result in the small claim limit for exponential utility which we will compare. On the other hand, we also turn our attention to the interesting case of comparing the power law utility approximation in the small claim limit with the exponential utility model from Chapter 4.

5.2.1 Comparison of Power Law Utility Model from the Small Claim Limit with the Exponential Utility Model from the Large Claim Limit

To compare a power law utility model with an exponential utility model, we have seen in Remark 2.0.1 that

$$\alpha = \frac{R}{x_0},$$

where α is the absolute risk aversion (parameter in exponential utility), R the relative risk aversion (parameter in power law utility) and x_0 is the initial wealth, has to hold true. But note that this is only a local assertion as the absolute risk aversion is constant for exponential utilities, but for power utilities, it changes over time. Hence this relationship definitely does not hold for long time horizons. But for our setting of $T=1$, this achieves the purpose.

We focus now on a scenario in the small claim limit and one in the large claim limit for a Put option.

5.2.1.1 Small Claim Limit

Parameters - We choose the following parameters:
$q=0.01,\ \varrho=0.8,\ K=100,\ T=1,\ t=0\ ,\mu=0.04,\ \eta=0.3,\ \sigma=0.35,\ \nu=\varrho\frac{\mu\eta}{\sigma}=0.0274,$
$R=3,\ x_0=500$

By the choice of those parameters, we get implicitly that $\alpha=\frac{R}{x_0}=0.006$ and that $\gamma=q\alpha(1-\varrho^2)=9.6\cdot 10^{-6}$. Moreover, note that we chose ν in such a way that the drift of Y_t under $\mathbb{Q}_{\min}$ is zero, i.e. $\delta=\nu-\varrho\frac{\mu\eta}{\sigma}=0$.

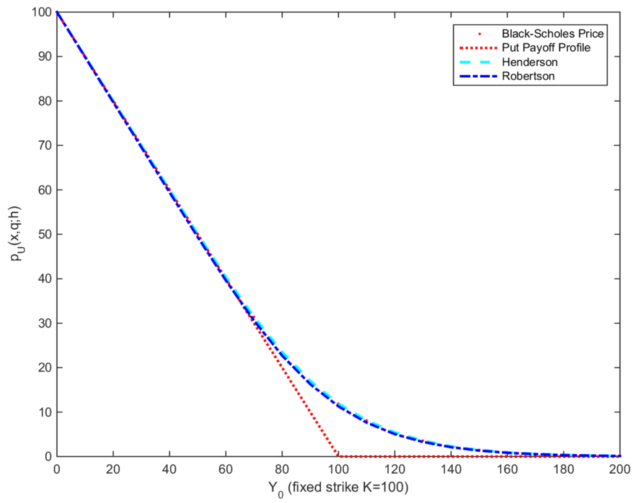
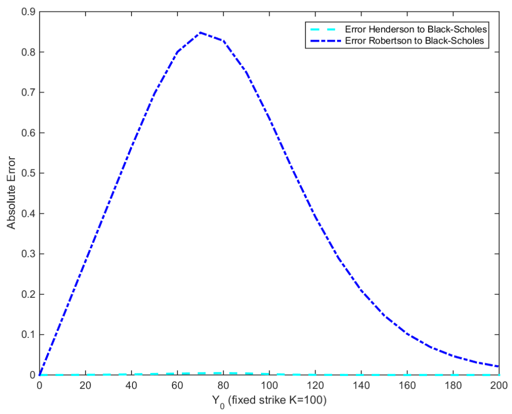
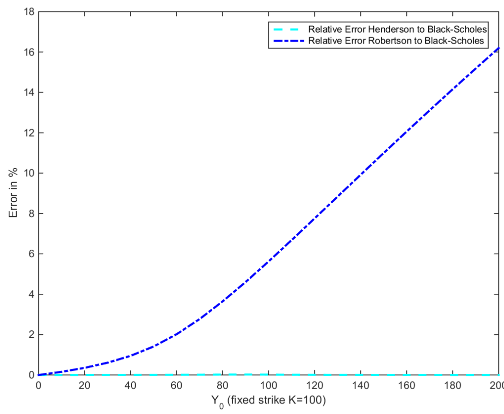


Figure 5.2.1: Comparison of the price for a Put option in the small claim limit.

We plotted in Figure 5.2.1 the two price dynamics with respect to the initial value Y_0.
'Henderson' represents the price dynamics given in Theorem 3.2.1, where the prices of 'Robertson' are from (4.3.1). For ease of convenience, we plotted the standard Put payoff profile (dotted line) and the Black-Scholes prices.
Surprisingly, they are all almost identical and there is only a marginal deviation from Black-Scholes prices seen by naked eye.
In Figure 5.2.2, we therefore compare the deviations of the two approaches with respect to the classical arbitrage-free Black-Scholes price $\mathbb{E}^{\mathbb{Q}}[h(Y_T)]$. As we already have seen in Chapter 3, Black-Scholes prices and 'Henderson' are almost identical due to the negligible impact of the second order term in the pricing formula from the small claim limit.



(a) Absolute error.



(b) Relative error.

Figure 5.2.2: Deviations from Black-Scholes price $\mathbb{E}^{\mathbb{Q}}[h(Y_T)]$.

It turns out that the large claim limit approach deviates significantly from Black-Scholes prices. Even though we did increase the number of grid points for the numerical approximation, the deviation did not change significantly hence we can exclude numerical errors as a source of deviation. Therefore, we can conclude that the deviation is due to the large position effect as seen in Chapter 4. We have seen in Figure 4.4.5 that the large claim limit model produces prices near to the arbitrage-free price for small claim quantities. We even have the convergence

$$\lim_{q \to 0} p_{U_\alpha}(x, q; h) = \mathbb{E}^{\mathbb{Q}}[h(Y_T)],$$

but the large position effect dominates quite fast, even for position sizes $q \approx 0.01$ and prices are pushed towards the minimal arbitrage-free price.
Of course, we will expect a different behavior for large claim positions. This is subject of our next paragraph.

5.2.1.2 Large Claim Limit

Parameters - We choose the following parameters:
$q = 100,\ \varrho = 0.8,\ K = 100,\ T = 1,\ t = 0\ , \nu = 0.0274,\ \mu = 0.04,\ \eta = 0.3,\ \sigma = 0.35,\ R = 3,$
$x_0 = 500$

Also here, we choose ν in such a way that the drift of Y_t is zero under $\mathbb{Q}_{\min}$.

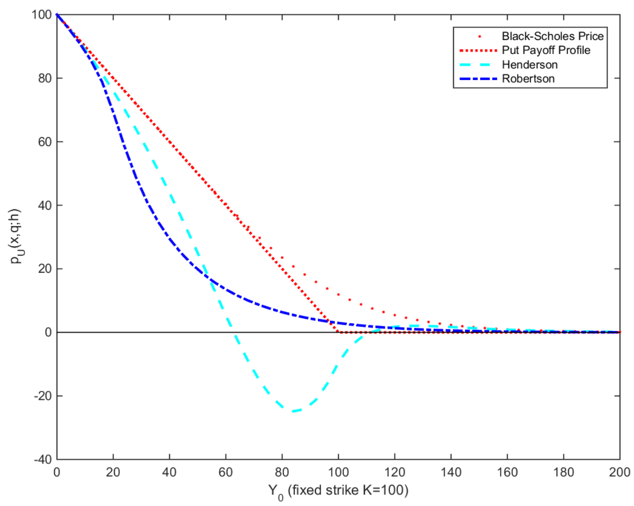
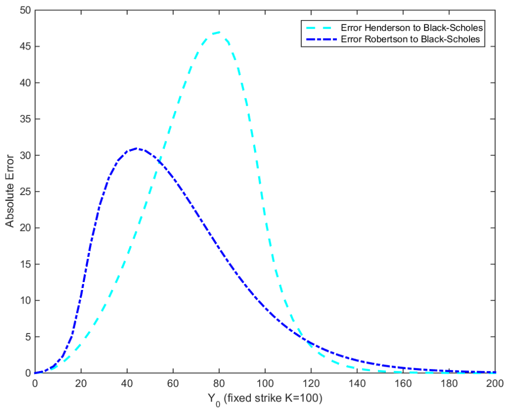
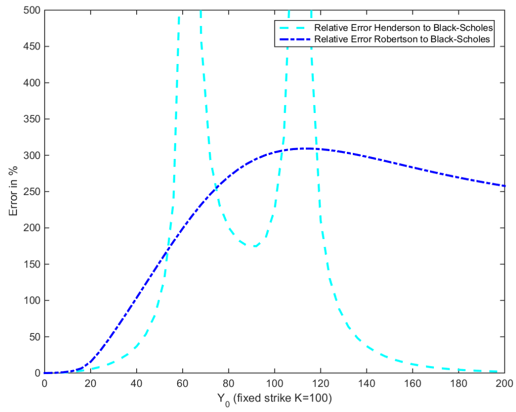


Figure 5.2.3: Comparison of the price for a Put option in the large claim limit.

Figure 5.2.3 shows a very different behavior. We see from the dynamics that 'Robertson' produces still reasonable prices when taking into account the large position effect as established and visualized in Figure 4.4.5.
In contrast to that, the approach by 'Henderson' provides us completely useless prices, which are even negative (hence not arbitrage-free) and not anymore applicable. The negative prices derive their origin in the dominating second order term (in the case of large positions).
The resulting errors are plotted in Figure 5.2.4. As mentioned, we see that both approaches do not depict the arbitrage-free Black-Scholes prices.



(a) Absolute errors.



(b) Relative errors.

Figure 5.2.4: Deviations from Black-Scholes price $\mathbb{E}^{\mathbb{Q}}[h(Y_T)]$.

Let us turn out attention to the comparison of the two models under exponential utility, where we do not use the local argument of $\alpha = \frac{R}{x_0}$ anymore.

5.2.2 Comparison of Exponential Utility Model from the Small Claim Limit with the Exponential Utility Model from the Large Claim Limit

We shall now compare the exponential utility model from the large claim approach with the one derived in the small claim limit.

5.2.2.1 Small Claim Limit

Parameters - We choose the following parameters:
$q = 0.01,\ \varrho = 0.8,\ K = 100,\ T = 1,\ t = 0,\ \nu = 0.0274,\ \mu = 0.04,\ \eta = 0.3,\ \sigma = 0.35,$
$R = 3,\ x_0 = 500$

Due to the fact that the exponential utility model from the small claim limit provides almost identical prices as the power utility model from the small claim limit approach. We have seen some considerable differences in the magnitude of the second order term like we have investigated in Section 3.3.2.2, but as the second order term is anyway almost negligible, we discover nearly the same plots as in Section 5.2.1.1.
Hence, also in this case, both models depict the Black-Scholes prices in a very exact manner.

But from a computational point of view, the exponential model derived in the small claim limit is much more efficient as it requires only the numerical approximation of the conditional variance of $h(Y_T)$ - we will discuss the computational effort in the next section.

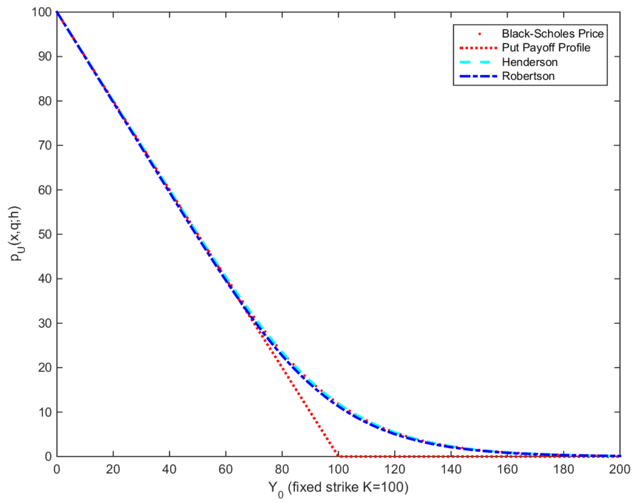
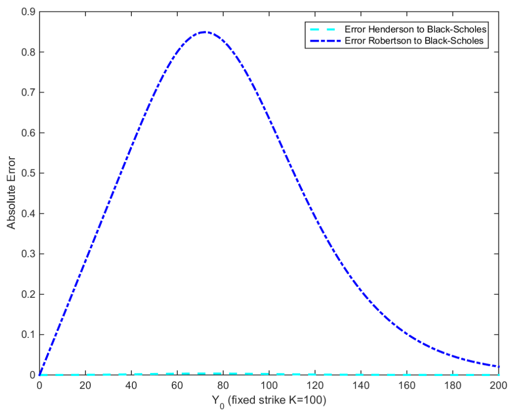
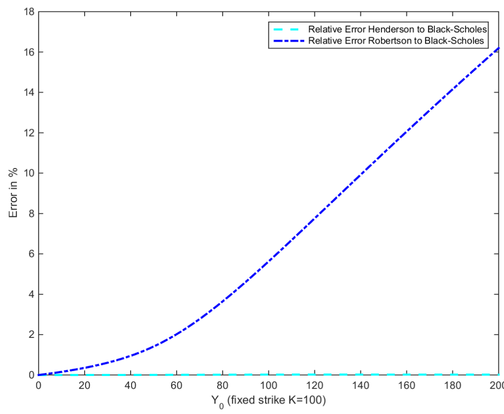


Figure 5.2.5: Comparison of the price for a Put option in the small claim limit.



(a) Absolute errors.



(b) Relative errors.

Figure 5.2.6: Deviations from Black-Scholes price $\mathbb{E}^{\mathbb{Q}}[h(Y_T)]$.

5.2.2.2 Large Claim Limit

Parameters - We choose the following parameters:
$q = 100,\ \varrho = 0.8,\ K = 100,\ T = 1,\ t = 0\ , \nu = 0.0274,\ \mu = 0.04,\ \eta = 0.3,\ \sigma = 0.35,$
$R = 3,\ x_0 = 500$

Lastly, we compare the two exponential utility models in the large claim limit. Again, we recover a similar behavior as seen in Section 5.2.1.2. 'Henderson' provides us with completely useless prices (the negative prices come from the large weighting of the second order term) while 'Robertson' depict the large position effect.

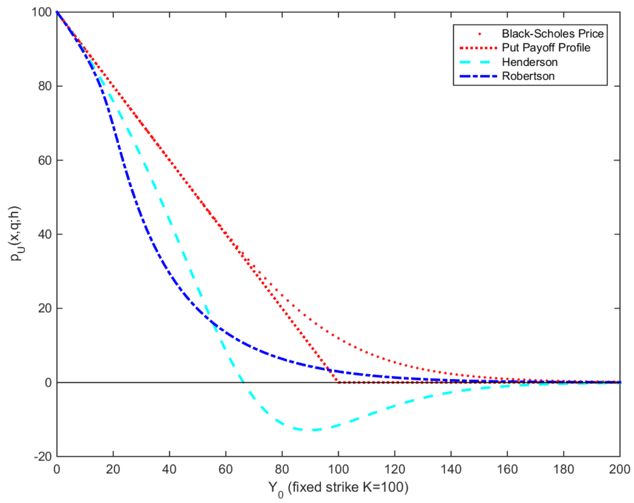
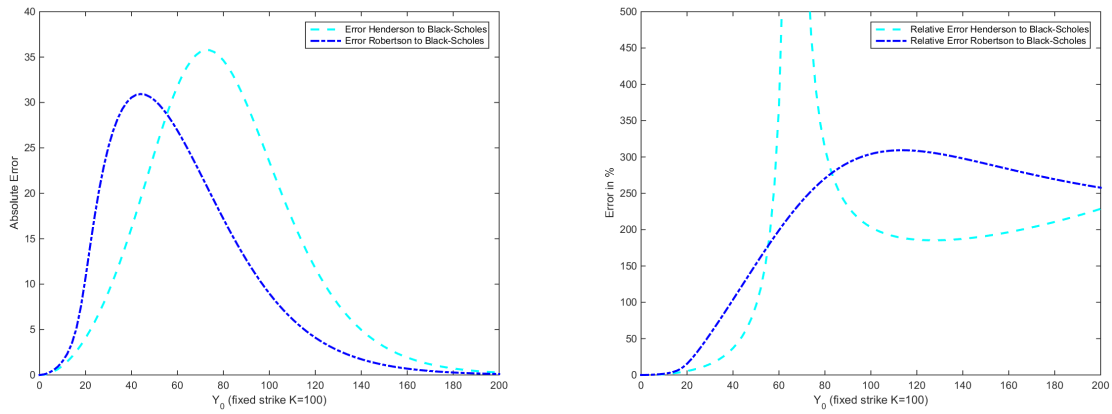


Figure 5.2.7: Comparison of the price for a Put option in the large claim limit.



(a) Absolute errors.

(b) Relative errors.

Figure 5.2.8: Different errors with respect to the arbitrage-free price $\mathbb{E}^{\mathbb{Q}}[h(Y_T)]$.

5.2.3 Computational Resources

We have seen, that the two models perform very well with reasonable prices in the respective claim limit. In addition to that, we also pointed out that the large claim approach by 'Robertson' provides us with prices nearly consistent with 'Henderson' in the (very) small claim limit. But already with little increasing claim sizes, the large position effect shows up and the spread with respect to Black-Scholes prices increases.

Lastly, we address the question of performance and computational resources. modeling the value function resp. average utility indifference prices under power utility derived by 'Henderson' requires the numerical approximation of double Riemann integrals with stochastic integrands. In our implementation, we tackled this problem with a by `MatLab` provided function (`dblquad`) at each point in a discretization of the time and space interval. This of course requires a lot of computational resources and, in our case, could take

up to hours of computation. Certainly, one could devote a lot of time for optimizing above approach but this was not our focus in this thesis.
In contrast to that, the exponential utility approach by 'Henderson' requires just the modeling of the conditional variance which can be implemented very easily and efficient. This took us time in the area of several seconds.
For modeling prices from 'Robertson', one has to numerically approximate an integral (we did this by Gauss quadrature), also an easy and tractable challenge requiring several seconds for computation.
In conclusion, this is also one of the main advantages of working under exponential utility: calculations become more tractable mathematically as well as from a computational point of view.

5.3 Conclusion

In conclusion, the small claim limit approach from Chapter 3 provides prices close to the arbitrage-free Black-Scholes price for very small position sizes. Nevertheless, also the large claim limit model gives us feasible prices in this limit, but the large position effect shows up very fast (already for $q = 1$). On the other hand side, the small claim limit approach is not at all applicable for large position sizes as expected. However, the large claim limit approach gives us a more robust tool in the sense of applicability with the drawback of larger deviations from the arbitrage-free price - which does not have to be a disadvantage, as we are in a setting of incomplete markets and prices are not unique.
Developing a theory from a model also involves numerous disadvantages which of course can be criticized. A big drawback of both approaches is that they are based on mathematical assumptions. This yields in restrictions as seen for example in Chapter 3, where we were not able to price short Call options.
Moreover, a general disadvantage of the approach of pricing claims based on individual utilities is that explicit calculations are possible rather rarely (most likely in the case of exponential utility) and one has to study their approximations in some limit. Therefore, often only 'local' statements are achievable. Hence it seems like exponential utility is very suitable in such a consensus. Unlikely, this is not the whole truth: for example, it turns out that utility indifference prices are independent of the initial wealth x_0, which could be undesirable as one could imagine that initial wealth has a significant impact on the risk aversion and hence prices.
Despite this, the utility indifference approach provides an accepted methodology for pricing claims under market incompleteness. Moreover, the goal of this approach is to find optimal investment strategies and by this one obtains directly hedging strategies. Thus, pricing and hedging can be viewed as the same problem, which is of course a big advantage.
Lastly, we point out that Chapter 4 provides an approach for a sequence of markets becoming complete by letting the position size resp. the correlation become infinity resp. one. A drawback is the lack of room for interpretation. The limiting market has no plausible economical interpretation and only assertions and studies in a large, but fixed market in the sequence are plausible. But it gives an enormous insight in the drivers for the limiting prices, depending on the speed of convergence.

Chapter 6

Small Claim Limit for General Semimartingale Models

We provide here a heuristic overview on a general semimartingale model in the small claim limit. Our main references are [Kal09] and [MK12].

Let S_t denote the price process of a traded asset in the market $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{F}, \mathbb{P})$ for a finite time horizon $T > 0$. Moreover, let Y_t denote another price process of an nontraded asset on which an European claim h is written, denoted by $h(Y_T)$. We assume a constant numeraire given by the bank account $B_t = 1$ and that S_t as well as Y_t are semimartingales.
The agent's goal is to maximize her expected utility from terminal wealth given she possesses an initial wealth of $x_0 > 0$. This means that the investor tries to maximize

$$\mathbb{E}^{\mathbb{P}}\left[U(x_0 + (\pi \cdot S)_T + qh(Y_T)\right]$$

over all admissible strategies $\pi \in \mathcal{H}$.
If the agent has the choice of exchanging n units of $h(Y_T)$ for a premium $p = p(n)$ per unit, she would accept the offer in the case if this trade will raise her (expected) utility at time T which then would be $\sup_{H\in\mathcal{H}} \mathbb{E}^{\mathbb{P}}\left[U(x_0 + np + (\pi \cdot S)_T + q'h(Y_T))\right]$, for $q' = q - n < q$ if $p > 0$.
Otherwise, she would deny the offering and her value function stays the same. The lowest price for which the agent would accept the offer is the average utility indifference price $p_U(x, q; h)$. Due to the fact that this premium is very hard to determine, we try to expand it in the neighborhood of $(x, 0; h)$.
For this, we rely on the following assumption:

Assumption 7. We assume a smooth dependence with respect to q

$$p(x_0, q; h) = p(0) + \delta q + o(q) \tag{6.0.1}$$

for some constants $p(0), \delta$.
Moreover, for $\pi \in \mathcal{H}$

$$\pi(x_0, q; h) = \pi^\star + \eta q + o(q) \tag{6.0.2}$$

for constants $\pi^\star$ denoting the optimal strategy in the classical Black-Scholes-Merton problem and constant

η denoting the hedge in $h(Y_T)$.

Our goal is now to find these constants:

6.1 Optimal Strategy without Contingent Claims

In a first step, consider the classical Black-Scholes-Merton investment problem (without any claim). We present here different approaches of finding the optimal strategy.
By the definition of the optimal strategy $\pi^\star$, we should have

$$\mathbb{E}^{\mathbb{P}}\left[U(x_0 + ((\pi^\star + \pi') \cdot S)_T)\right] \leq \mathbb{E}^{\mathbb{P}}\left[U(x_0 + (\pi^\star \cdot S)_T)\right]$$

for any $\pi' \in \mathcal{H}$. Applying Taylor expansion to the left-hand side leads us to

$$\mathbb{E}^{\mathbb{P}}\left[U(x_0 + ((\pi^\star + \pi') \cdot S)_T)\right] \approx \mathbb{E}^{\mathbb{P}}\left[U(x_0 + (\pi^\star \cdot S)_T)\right] + \mathbb{E}^{\mathbb{P}}\left[U'(x_0 + (\pi^\star \cdot S)_t)(\pi' \cdot S)_T\right].$$

We define a new probability measure $\tilde{\mathbb{Q}}$ by

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} := \frac{U'(x_0 + (\pi^\star \cdot S)_T)}{\mathbb{E}^{\mathbb{P}}\left[U'(x_0 + (\pi^\star \cdot S)_T)\right]}.$$

We find that $\mathbb{E}^{\mathbb{P}}\left[U(x_0 + ((\pi^\star + \pi') \cdot S)_T)\right]$ is dominated by $\mathbb{E}^{\mathbb{P}}\left[U(x_0 + (\pi^\star \cdot S)_T)\right]$ if and only if $\mathbb{E}^{\tilde{\mathbb{Q}}}\left[(\pi' \cdot S)_T\right] \leq 0$ for all $\pi' \in \mathcal{H}$. The latter holds if and only if S is a $\tilde{\mathbb{Q}}$-martingale.
In summary, we have found that an arbitrary element $\pi^\star \in \mathcal{H}$ maximizes the expected utility at time T if and only if $\tilde{\mathbb{Q}}$ is an equivalent martingale measure for S.
An equivalent and maybe more straightforward approach is the following: As we try to maximize over $\pi^\star \in \mathcal{H}$

$$\mathbb{E}^{\mathbb{P}}\left[U\left(x_0 + \int_0^T \pi_t^\star dS_t\right)\right],$$

we get the first order condition of

$$\mathbb{E}^{\mathbb{P}}\left[U'\left(x_0 + \int_0^T \pi_t^\star dS_t\right)(S_T - S_0)\right] \stackrel{!}{=} 0.$$

Hence, by a change of measure $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}$ as above, we obtain

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\left[S_T - S_0\right] \stackrel{!}{=} 0,$$

which is tantamount to the martingale property of S_t under $\mathbb{Q}$.
Lastly, we also want to provide an approach derived by considering the dual problem. We have seen in (2.1.1) that the dual approach gives us an upper bound in terms of the generalized relative entropy. Essentially for any equivalent martingale measure $\tilde{\mathbb{Q}}$ and any $\pi^\star \in \mathcal{H}_{\text{perm}}$, we have

$$\mathbb{E}^{\mathbb{P}}\left[U\left(x_0 + \int_0^T \pi_t^\star dS_t\right)\right] \leq \mathbb{E}^{\mathbb{P}}\left[V\left(y\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\right)\right] + xy.$$

This inequality is an equality, if we choose $y\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}$ in the optimal way, i.e. if $y\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} = U'(x_0 + \int_0^T \pi_t^\star dS_t)$ and if $\mathbb{E}^{\mathbb{P}}[(x + \int_0^T \pi_t^\star dS_t) y \frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}] = xy$.
Hence we have shown that an arbitrary strategy $\hat{\pi}$ optimizes the primal problem (in the set of $\mathcal{H}_{\text{perm}}$) if ([MK12, Proposition 2.2])

1. $U'(x_0 + \int_0^T \hat{\pi}_t dS_t) = y\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}$ for an equivalent martingale measure $\bar{\mathbb{Q}}$, and

2. $x = \mathbb{E}^{\mathbb{P}}[(x + \int_0^T \hat{\pi}_t dS_t)\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}] = \mathbb{E}^{\mathbb{P}}[(U')^{-1}(y\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}})\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}]$.

6.2 Optimal Strategy including Contingent Claims

Having given several criteria for an optimizing strategy in the investment problem without any claim, we no want to turn our attention to the optimization problem including q claims $h(Y_T)$ with price $p = p(x_0, q; h)$ per unit. As already seen, we try to maximize

$$\begin{aligned}(6.2.1)\quad & \mathbb{E}^{\mathbb{P}}\left[U(x_0 + qp + (\pi(x_0, q; h)\cdot S)_T - qh(Y_T))\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[U\left(x_0 + (\pi^\star \cdot S)_T + q(p(0) + \delta q + ((\eta + o(1))\cdot S)_T - h(Y_T)) + o(q^2)\right)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[U\left(x_0 + (\pi^\star \cdot S)_T\right)\right] + q\mathbb{E}^{\mathbb{P}}\left[U'\left(x_0 + (\pi^\star \cdot S)_T\right)(p(0) + \delta q + ((\eta + o(1))\cdot S)_T - h(Y_T)\right] \\ &+ \frac{q^2}{2}\mathbb{E}^{\mathbb{P}}\left[U''\left(x_0 + (\pi^\star \cdot S)_T\right)(p(0) + (\eta\cdot S)_T - h(Y_T))^2\right] + o(q^2)\end{aligned}$$

over all $\pi \in \mathcal{H}$.
As we have in the case of power law utility $U(x) = \frac{1}{1-R}x^{1-R}$ that $U''(x) = -R\frac{U'(x)}{x}$, this reduces to

$$\begin{aligned}(6.2.2)\qquad & \mathbb{E}^{\mathbb{P}}\left[U\left(x_0 + (\pi^\star \cdot S)_T\right)\right] + q\mathbb{E}^{\mathbb{P}}\left[U'\left(x_0 + (\pi^\star \cdot S)_T\right)\right]\mathbb{E}^{\tilde{\mathbb{Q}}}\left[p(0) + \delta q + ((\eta + o(1))\cdot S)_T - h(Y_T)\right] \\ & - q^2\mathbb{E}^{\mathbb{P}}\left[U'(x_0 + (\pi^\star \cdot S)_T)\right]\frac{R}{2}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\frac{x_0 + (\pi^\star \cdot S)_T}{x_0^2}\left(\frac{p(0) + (\eta\cdot S)_T - h(Y_T)}{x_0^{-1}(x_0 + (\pi^\star \cdot S)_T)}\right)^2\right] \\ & + o(q^2).\end{aligned}$$

As we are working with the optimizer $\pi^\star$, we know that $\tilde{\mathbb{Q}}$ is an equivalent martingale measure, hence

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\left[((\eta + o(1))\cdot S)_T\right] = 0.$$

We define a second probability measure $\bar{\mathbb{Q}}$ equivalent to $\tilde{\mathbb{Q}}$ by

$$\frac{d\bar{\mathbb{Q}}}{d\tilde{\mathbb{Q}}} = \frac{x_0 + (\pi^\star \cdot S)_T}{x_0} = \frac{x_0 + (\pi^\star \cdot S)_T}{\mathbb{E}^{\tilde{\mathbb{Q}}}\left[x_0 + (\pi^\star \cdot S)_T\right]}.$$

Moreover, it follows that $\bar{\mathbb{Q}}$ is an equivalent martingale measure for the numeraire $N_t = \frac{x_0 + (\pi^\star \cdot S)_t}{\mathbb{E}^{\tilde{\mathbb{Q}}}[x_0 + (\pi^\star \cdot S)_T]}$.
Hence

$$\frac{S_t}{N_t} = \frac{S_t\mathbb{E}^{\tilde{\mathbb{Q}}}\left[x_0 + (\pi^\star \cdot S)_t\right]}{x_0 + (\pi^\star \cdot S)_t} = \frac{S_t x_0}{x_0 + (\pi^\star \cdot S)_t}$$

is a $\bar{\mathbb{Q}}$-martingale.
In the sequel, we denote by $\bar{p}$, $\bar{h}$, $\bar{S}$ resp. $\bar{\pi}$ the discounted prices, claim payoffs, stock price resp. hedging strategy with respect to the numeraire N_t. As $\pi \in \mathcal{H}$ is, by definition, self-financing, so is η, hence the

following representation holds

$$\frac{p(0)+(\eta\cdot S)_T-h(Y_T)}{x_0^{-1}(x_0+(\pi^\star\cdot S)_T)}=\bar{p}(0)+(\eta\cdot\bar{S})_T-\bar{h}(Y_T). \tag{6.2.3}$$

It follows that (6.2.2) reduces to

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[U(x_0+(\pi^\star\cdot S)_T)\right]+q\mathbb{E}^{\mathbb{P}}\left[U'(x_0+(\pi^\star\cdot S)_T)\right]\Big(p(0)-\mathbb{E}^{\tilde{\mathbb{Q}}}\left[h(Y_T)\right]\Big)\\ +q^2E^{\mathbb{P}}\left[U'(x_0+(\pi^\star\cdot S)_T)\right]\left(\delta-\frac{R}{2x_0}\varepsilon^2(\eta)\right)\\ +o(q^2),\end{aligned} \tag{6.2.4}$$

where

$$\varepsilon^2(\eta):=\mathbb{E}^{\bar{\mathbb{Q}}}\left[(\bar{p}(0)+(\eta\cdot\bar{S})_T-\bar{h}(Y_T))^2\right]. \tag{6.2.5}$$

Note that we initially wanted to optimize (6.2.1). By our assumption to $\pi\in\mathcal{H}$, this reduces to an optimization over $\eta\in\mathcal{H}$. Hence, we have to minimize (6.2.5) to find a maximizer for (6.2.4).
This *quadratic* minimizing problem has a nice interpretation, which we will establish in the next paragraph.

6.2.1 *Quadratic* Hedging

The goal is to find the strategy $\eta^\star\in\mathcal{H}$ which minimizes

$$\varepsilon^2(\eta)=\mathbb{E}^{\mathbb{Q}}\left[(x_0+(\eta\cdot S)_T-h(Y_T))^2\right],$$

where the expectation is taken under some probability measure, say, $\mathbb{Q}$.
The term $x_0+(\eta\cdot S)_T$ is the value of the replicating portfolio, as η is self-financing and hence $\varepsilon^2(\eta)$ measures the mean squared error of replicating the claim h with the strategy η. In a complete market framework, this error is of course zero due to the possibility of perfect replication. In all other cases, under any market incompleteness, we have seen that the approach by replicating the random payoff h does not work anymore, hence investors are always exposed to some risk and $\varepsilon^2(\eta)$ does not vanish and a possibility to measure this unhedgeable risk is exactly given by $\varepsilon^2(\eta)$.
In the complete market, the unique arbitrage-free price is given by

$$\mathbb{E}^{\mathbb{Q}}[h(Y_T)],$$

while there is no unique price under market incompleteness.
Under incompleteness, a reasonable suggestion for the price might therefore be

$$\mathbb{E}^{\mathbb{Q}}[h(Y_T)]+\xi\varepsilon^2(\eta^\star),$$

where ξ is a parameter representing the risk-aversion of a certain agent (the higher ξ, the more risk-averse the investor), hence the price would consist of the unique arbitrage-free price plus resp. minus a compensation for the unhedgeable risk.
We have therefore to determine $\eta^\star$ and $\varepsilon^2(\eta^\star)$.

In the case, where S_t is a $\mathbb{Q}$-martingale[20], **Galtchouk-Kunita-Watanabe decomposition** provides us a way of determining these quantities.

Proposition 6.2.1. ([Sch99a, (0.3)])
Any square $\mathbb{Q}$-integrable random variable H can uniquely be written as

$$H = \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_0] + \int_0^T \chi_u^H dS_u + L_T^H \quad \mathbb{P}\text{-}a.s.,$$

where L_t^H is a martingale strongly orthogonal to the martingale S_u and χ_u^H a predictable process.

Hence, denote by $V_t = \mathbb{E}^{\mathbb{Q}}[h(Y_T)|\mathcal{F}_t]$ the martingale generated by h. Now, Galtchouk-Kunita-Watanabe decomposition applied to V_t yields a representation of the form

$$V_t = \mathbb{E}^{\mathbb{Q}}[h(Y_T)|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[h(Y_T)|\mathcal{F}_0] + (\chi \cdot S)_t + L_t,$$

where L_t is a $\mathbb{Q}$-martingale orthogonal to S_t and χ a predictable process. Here, we require that S_t is a $\mathbb{Q}$-martingale.
We can identify, using the uniqueness of the Galtchouk-Kunita-Watanabe decomposition that

$$\eta^\star = \chi$$

and

$$\varepsilon^2(\eta^\star) = \mathbb{E}^{\mathbb{Q}}\left[(V_T - \mathbb{E}^{\mathbb{Q}}[h(Y_T)] - (\chi \cdot S)_T)^2\right] = \mathbb{E}^{\mathbb{Q}}\left[L_T^2\right] = \mathbb{E}^{\mathbb{Q}}[\langle V, V - (\eta^\star \cdot S)\rangle_T],$$

where we used in the last equality the fact that $M_t^2 - \langle M\rangle_t$ is a martingale for a martingale M_t and the fact that $\langle V - (\eta^\star \cdot S), V - (\eta^\star \cdot S)\rangle = \langle V, V - (\eta^\star \cdot S)\rangle$. More concretely, we can write

$$\chi_t = \frac{d\langle V, S\rangle_t}{d\langle S, S\rangle_t}.$$

Summarizing, Galtchouk-Kunita-Watanabe decomposition gives us a way of finding risk-minimizing strategies in the case where S_t is a martingale.

Turning our attention back to (6.2.4):
We have to find the minimizing $\eta^\star$ of

$$\mathbb{E}^{\bar{\mathbb{Q}}}\left[(\bar{p}(0) + (\eta \cdot \bar{S})_T - \bar{h})^2\right]. \tag{6.2.6}$$

By neglecting the $o(q^2)$ terms, we see that η is the integrand in the GKW decomposition of the $\tilde{\mathbb{Q}}$-martingale $\bar{V}_t = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\bar{h}(Y_T)|\mathcal{F}_t\right]$, that is

$$\eta_t^\star = \frac{d\langle \bar{V}, \bar{S}\rangle_t^{\bar{\mathbb{Q}}}}{d\langle \bar{S}, \bar{S}\rangle_t^{\bar{\mathbb{Q}}}}.$$

Together with the known form of $\tilde{\pi}^\star = \frac{\mu}{\sigma^2 R} X$, we have found the parameters in (6.0.2). Turning our attention to the price, we are left with finding $p(0)$ and δ in (6.0.1).

[20] It turns out, that other cases are also tractable, but more involved. They still heavily rely on the Galtchouk-Kunita-Watanabe resp. Föllmer-Schweizer decomposition, which is the Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure [Sch99a].

By the definition of the average utility indifference price, the following has to hold

$$\mathbb{E}^{\mathbb{P}}\left[U(x_0 + (\pi^\star \cdot S)_T)\right] \overset{!}{=} \mathbb{E}^{\mathbb{P}}\left[U(x_0 + qp + (\pi^\star \cdot S)_T - qh(Y_T))\right],$$

which implies, using (6.2.4), that

$$p(0) = \mathbb{E}^{\tilde{\mathbb{Q}}}[h(Y_T)] \text{ and } \delta = \frac{R}{2x_0}\varepsilon^2(\eta).$$

In conclusion, we have shown that

$$\pi(q) = \pi^\star + q\eta + o(n),$$

where $\pi^\star$ is the optimizing strategy if and only if S_t is a $\tilde{\mathbb{Q}}$-martingale and

$$\eta_t^\star = \frac{d\langle \bar{V}, \bar{S}\rangle_t^{\bar{\mathbb{Q}}}}{d\langle \bar{S}, \bar{S}\rangle_t^{\bar{\mathbb{Q}}}}.$$

Moreover, we have identified in

$$p(x, q; h) = p(0) + q\delta + o(n)$$

the parameters

$$p(0) = \mathbb{E}^{\tilde{\mathbb{Q}}}[h(Y_T)] \text{ and } \delta = \frac{R}{2x_0}\varepsilon^2(\eta).$$

Hence the price consists of a first order term coinciding with the Black-Scholes price and an additional term that compensates the unhedgeable risk per unit.

6.3 Comparison with Small Claim Limit Approach

We have seen in the beginning of Chapter 3, that the Merton hedging strategy of investing in S_t with initial wealth of $X_t + qC_t$ is given by

$$\tilde{\pi}_t = \frac{\mu}{\sigma^2 R}(X_t + qC_t).$$

Moreover, by considering $\tilde{\mathbb{Q}}$, we see that

$$\begin{aligned}\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} &:= \frac{U'(x_0 + (\pi^\star \cdot S)_T)}{\mathbb{E}^{\mathbb{P}}\left[U'(x_0 + (\pi^\star \cdot S)_T)\right]} = \frac{\exp\left(-\frac{\mu^2}{\sigma^2}T + \frac{\mu^2}{2\sigma^2 R}T - \frac{\mu}{\sigma}B_T\right)}{\exp\left(-\frac{\mu^2}{2\sigma^2}T + \frac{\mu^2}{2\sigma^2 R}T\right)} \\ &= \exp\left(-\frac{\mu}{\sigma}B_T - \frac{\mu^2}{2\sigma^2}T\right) = \mathcal{E}\left(-\frac{\mu}{\sigma}\cdot B\right)_T,\end{aligned}$$

which is nothing else than the minimal martingale measure $\mathbb{Q}_{\min}$.
Lastly,

$$\frac{d\bar{\mathbb{Q}}}{d\tilde{\mathbb{Q}}} = \frac{x_0 + (\pi^\star \cdot S)_t}{x_0} = \exp\left(\frac{\mu}{\sigma R}\tilde{B}_T - \frac{\mu^2}{2\sigma^2 R^2}T\right) = \mathcal{E}\left(\frac{\mu}{\sigma R}\cdot\tilde{B}\right)_T$$

for a $\mathbb{Q}_{\min}$-Brownian motion $\tilde{B}_t$. Therefore, by Yor's formula

$$\begin{aligned}\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} &= \frac{d\bar{\mathbb{Q}}}{d\tilde{\mathbb{Q}}}\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = \mathcal{E}\left(\frac{\mu}{\sigma R}\cdot\tilde{B}\right)_T \mathcal{E}\left(-\frac{\mu}{\sigma}\cdot B\right)_T = \mathcal{E}\left(\frac{\mu}{\sigma R}\cdot\tilde{B} - \frac{\mu}{\sigma}\cdot B\right)_T \exp\left(\langle\frac{\mu}{\sigma R}\cdot\tilde{B}, -\frac{\mu}{\sigma}\cdot B\rangle_T\right)\\ &= \mathcal{E}\left(\frac{\sigma(1-R)}{\mu R}\cdot B\right)_T.\end{aligned}$$

We then see that $\bar{\mathbb{Q}}$ coincides with the measure $\hat{\mathbb{P}}$ defined in Theorem 3.2.1.
As S_t is a $\mathbb{Q}_{\min}$-martingale, it follows that $\bar{\mathbb{Q}}$ is an equivalent martingale measure for the numeraire $N_t = \exp\left(\frac{\mu}{\sigma R}\tilde{B}_T - \frac{\mu^2}{2\sigma^2 R^2}T\right)^{-1}$.
By Bayes formula, we get

$$\begin{aligned}\bar{V}_t = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\bar{h}(Y_T)|\mathcal{F}_t\right] &= \left(\frac{d\bar{\mathbb{Q}}}{d\tilde{\mathbb{Q}}}|_{\mathcal{F}_t}\right)^{-1}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\frac{d\bar{\mathbb{Q}}}{d\tilde{\mathbb{Q}}}|_{\mathcal{F}_T}\bar{h}(Y_T)|\mathcal{F}_t\right]\\ &= \frac{x_0}{x_0+(\pi^\star\cdot S)_t}\mathbb{E}^{\mathbb{Q}_{\min}}\left[h(Y_T)|\mathcal{F}_t\right]\end{aligned}$$

and

$$\bar{S}_t = S_t\frac{x_0}{x_0+(\pi^\star\cdot S)_t}.$$

Therefore, by setting $C_t = \mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t], C_t^Y = \partial_Y C_t$

$$\begin{aligned}\left(\frac{x_0}{x_0+(\pi^\star\cdot S)_t}\right)^{-2} d\langle\bar{V},\bar{S}\rangle_t &= d\langle C,S\rangle_t = d\langle C^Y\eta Y\,dZ, \sigma S dB\rangle_t\\ &= \eta C_t^Y Y_t\sigma S_t\varrho dt.\end{aligned}$$

In the same way we get

$$\left(\frac{x_0}{x_0+(\pi^\star\cdot S)_t}\right)^{-2} d\langle\bar{S},\bar{S}\rangle = \sigma^2 S_t^2 dt$$

which leads to

$$\eta_t^\star = \frac{d\langle\bar{V},\bar{S}\rangle_t}{d\langle\bar{S},\bar{S}\rangle_t} = \frac{\eta\varrho}{\sigma}C_t^Y Y_t\frac{1}{S_t},$$

as desired under the observation that $\tilde{\pi}^\star = \tilde{\pi}(x_0,q;h) - \tilde{\eta}^\star$. This is exactly what Theorem 3.2.1 states. Turning our attention to the pricing formula, we see that the first term $\mathbb{E}^{\mathbb{Q}_{\min}}[h(Y_T)|\mathcal{F}_t]$ is derived easily. For the second term, by setting $\bar{C}_t := \mathbb{E}^{\bar{\mathbb{Q}}}[\bar{h}(Y_T)|\mathcal{F}_t]$ and $\bar{C}_t^Y := \partial_Y\bar{C}_t$, we obtain

$$\begin{aligned}\delta = \frac{R}{2x_0}\varepsilon^2(\eta^\star) &= \frac{R}{2x_0}\mathbb{E}^{\bar{\mathbb{Q}}}[(\bar{p}(0)+(\eta^\star\cdot\bar{S})_T - \bar{h}(Y_T))^2]\\ &= \frac{R}{2x_0}\mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_t^T d\langle C^Y dY, C^Y dY - \eta^\star d\bar{S}\rangle_u\right]\\ &= \frac{R}{2x_0}\mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_t^T\frac{(C_u^Y)^2(Y_u)^2\eta^2 - (C_u^Y)^2(Y_u)^2\eta^2\varrho^2}{\exp(\frac{\mu^2}{\sigma^2 R}u - \frac{\mu^2}{2\sigma^2R^2}u + \frac{\mu}{\sigma R}B_u)^2}du\right]\\ &= \frac{R}{2x_0}\eta^2(1-\varrho^2)\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_t^T\frac{(C_u^Y Y_u)^2}{(X_t^0/X_t)^2}du\right].\end{aligned}$$

In conclusion, the general semimartingale model by [Kal09] entirely reinforces to the basis risk model as presented in Chapter 3.

Finally, we address the question on the additional insight we gained by following this general approach. This general approach gave us a deeper understanding of the additional term in the value function (Theorem 3.2.1) and the average utility indifference price (Corollary 3.2.1). As we are dealing with an incomplete market framework, claims are not replicable and investors are exposed to unhedgeable risk. This risk must be quantified in order to speak about hedging resp. risk minimizing and our way of doing so was to consider the mean squared error. We then tried to minimize this error yielding in an optimal hedge.
To the end, we want to emphasize that the classical Black-Scholes hedging strategy does also minimize the mean squared error. This legitimates the use of this tool for quantifying error.

Appendix A

Fundamental Theorem of Asset Pricing

Definition A.0.1. Fix a physical measure $\mathbb{P}$. We then say that $\mathbb{Q}$ is an equivalent **(local) martingale measure** if $\mathbb{Q} \sim \mathbb{P}$ and all the (discounted) traded assets are $\mathbb{Q}$-(local) martingales.

The following theorem is known as First Fundamental Theorem of Asset Pricing. It relates the existence of a local martingale measure with the absence of arbitrage.

Theorem A.0.1. ([DS06, Theorem 8.2.1])
For a locally bounded semimartingale $S = (S_t)_{t \geq 0}$, the following assertions are equivalent:

- *There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which S a local martingale.*
- *S does not permit a free lunch with vanishing risk.*

From [DS06, Definition 11.2.2], we have that 'no free lunch with vanishing risk' implies 'no arbitrage'.
Now we give a short heuristic interpretation of the condition 'no free lunch with vanishing risk' and it's connection to 'no arbitrage.
An arbitrage opportunity is the existence of a trading strategy π_t such that $(\pi \cdot S)_T \geq 0$ and such that $\mathbb{P}[(\pi \cdot S)_T > 0] > 0$. In contrast to that, a 'free lunch' is the existence of a contingent claim $\bar{h} \geq 0, \bar{h} \neq 0$, which cannot be (super-)replicated by an admissible trading strategy π_t. But there exist claims $\bar{h}_i$ 'close to' $\bar{h}$ ($\lim_i \bar{h}_i \to \bar{h}$ in some sense/topology) which can be superreplicated by admissible trading strategies π_t^i - hence an agent may 'throw away'

$$(\pi^i \cdot S)_\infty - \bar{h}_i$$

for each i, yielding in a free lunch with vanishing risk. [FS10, p.9, p. 78, p.153].
Let us turn our attention to the Second Fundamental Theorem of Asset Pricing.

Theorem A.0.2. ([Fil09, Theorem 4.9])
Assume there exists an equivalent local martingale measure $\mathbb{Q}$. Then the following are equivalent:

- *The model is complete.*
- *The equivalent local martingale measure $\mathbb{Q}$ is unique.*

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